Math 118 Winter 2015 Lecture 19 (Feb. 5, 2015)

Midterm 1 Review 3

- Integration of functions involving roots.
  - $\int R\left(x, \left(\frac{a\,x+b}{c\,x+d}\right)^{1/m_1}, \ldots, \left(\frac{a\,x+b}{c\,x+d}\right)^{1/m_k}\right)$: Set $t = \left(\frac{a\,x+b}{c\,x+d}\right)^{1/m}$ where m is the least common multiple of $m_1, \ldots, m_k$.
  - $\int R(x, \sqrt{a\,x^2+b\,x+c})$: Reduce to the previous case or trivial case when $a\,x^2+b\,x+c=0$ has real roots; Set $t=\sqrt{a\,x^2+b\,x+c} \pm \sqrt{a}\,x$ when $a\,x^2+b\,x+c>0$ for all x.
  - Examples.

Example 1. Integrate the following.

a) $\displaystyle\int \frac{x^{1/2}\,\mathrm{d}x}{x^{3/4}+1}$;

b) $\displaystyle\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$;

c) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+2}}$.

Solution.

a) We have

$$\begin{aligned}
\int \frac{x^{1/2}\,\mathrm{d}x}{x^{3/4}+1} &\overset{t=x^{1/4}}{=\!=\!=} \int \frac{4\,t^5}{t^3+1}\,\mathrm{d}t \\
&= 4\int \left[t^2 - \frac{t^2}{t^3+1}\right]\mathrm{d}t \\
&= \frac{4}{3}t^3 - \frac{3}{4}\ln|t^3+1| + C \\
&= \frac{4}{3}\left[x^{3/4} - \ln\left|x^{3/4}+1\right|\right] + C. \qquad (1)
\end{aligned}$$

Exercise 1. Try $t = x^{3/4}+1$.[1]

b) We have

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x &\overset{t=\sqrt{x+4}}{=\!=\!=} \int \frac{2\,t^2}{t^2-4}\,\mathrm{d}t \\
&= \int \left[2 + \frac{8}{t^2-4}\right]\mathrm{d}t \\
&= 2\,t + 2\ln\left|\frac{t-2}{t+2}\right| + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C. \qquad (2)
\end{aligned}$$

c) We set $\sqrt{x^2+2} = x+t$ to obtain

$$x = \frac{2-t^2}{2\,t}, \qquad \sqrt{x^2+2} = \frac{2+t^2}{2\,t}, \qquad \mathrm{d}x = -\frac{2+t^2}{2\,t^2}\,\mathrm{d}t. \qquad (3)$$

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1. Thanks to Mr. Weicheng Li for observing this.

Thus

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2}} \overset{t=\sqrt{x^2+2}-x}{=\!=\!=\!=\!=} \int \frac{-\frac{2+t^2}{2\,t^2}\,\mathrm{d}t}{\frac{2+t^2}{2\,t}} = -\int \frac{\mathrm{d}t}{t} = -\ln|t|+C = -\ln\left|\sqrt{x^2+2}-x\right|+C. \qquad (4)$$

Exercise 2. Someone obtain $\ln\left|x+\sqrt{x^2+2}\right|+C$. Did he make a mistake? Explain.

Exercise 3. Calculate this integral using trig substitution.

Exercise 4. Calculate this integral using the change of variables indicated by Chebyshev's theorem.

- Chebyshev & Liouville
  - Chebyshev: Know how to integrate

    $$\int (a\,x+b)^r\,(c\,x+d)^s\,\mathrm{d}x \qquad (5)$$

    where $r, s \in \mathbb{Q}$.
  - Liouville: Know that if $\int f(x)\,e^{g(x)}\,\mathrm{d}x$, where f, g are rational, is elementary, then it must be of the form $R(x)\,e^{g(x)}$ where $R(x)$ is rational.
- Other issues and some challenge problems.
  - Simplification.

    When using trig substitutions we often need to represent one trig function by another trig function.

    Example 2. $x=\tan t$, then $\sin t = ?(x)$?

    Solution. We have

    $$x=\frac{\sin t}{\cos t}=\frac{\sin t}{\sqrt{1-\sin^2 t}} \Longrightarrow x^2\,(1-\sin^2 t)=\sin^2 t \Longrightarrow \sin t = \pm\frac{x}{\sqrt{1+x^2}}. \qquad (6)$$

    To determine the sign, we notice that in $x=\tan t$ we can take $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ which means $\sin t$ has the same sign as $\tan t$ and consequently we should take the plus sign.

    Remark 3. The issue seems a bit complicated as we could also take $t\in\left(\frac{\pi}{2},\frac{3\,\pi}{2}\right)$. However it seems to me for $\sin t$ to appear in the final answer, somewhere during the calculation $\cos t$ must present. As the original variable is x, $\cos t$ can only appear from $\sqrt{x^2+1}$ and we would have to choose a sign for it then. This sign would also depend on whether $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2},\frac{3\,\pi}{2}\right)$.

QUESTION 4. *(Problems like this will not be in the midterm; The meaning of $\frac{\partial f}{\partial a}$ is simply to differentiate treating a as the variable and ignoring x. You can also search "partial derivative" in wiki)*

Let $a_0\in\mathbb{R}$. Critique the following claim.

$$\int f(x,a)\,\mathrm{d}x = F(x,a)+C \Longrightarrow \int \left[\frac{\partial f}{\partial a}(x,a_0)\right]\mathrm{d}x = \frac{\partial F}{\partial a}(x,a_0). \qquad (7)$$

If you think it is true, prove it through definition of indefinite integrals, and apply it to calculate $\displaystyle\int \frac{\mathrm{d}x}{(x^2+1)^k}$ *with* $k=2,3$. *If you think it is false find a counter-example.*