Math 118 Winter 2015 Lecture 18 (Feb. 4, 2015)

Midterm 1 Review 2

Note. Hermite's method is not required.

- Integration of rational functions.
  - Method of partial fractions.

$$\frac{P(x)}{Q(x)} = P_0(x) + \sum \frac{A}{(x-a)^k} + \sum \frac{B\,x + C}{[(x+a)^2 + b^2]^k}. \tag{1}$$

i. If $\deg P \geqslant \deg Q$, then calculate

$$P = Q\,P_0 + R \tag{2}$$

where $\deg R < \deg Q$. If $\deg P < \deg Q$ then $P_0 = 0$ and $R = P$.

ii. Factorize Q.

iii. Write

$$\frac{R(x)}{Q(x)} = \sum \frac{A}{(x-a)^k} + \sum \frac{B\,x + C}{[(x+a)^2 + b^2]^k}. \tag{3}$$

iv. Integrate.

  - How to factorize. To factorize $Q(x) = a_n\,x^n + \cdots + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, check whether any $\frac{p}{q}$ with $p|\,a_0$, $q|\,a_n$ solves $Q(x) = 0$. If there is such $\frac{p}{q}$ then $Q(x)$ has a factor $\left(x - \frac{p}{q}\right)$. If there is no such $\frac{p}{q}$, try other ad hoc factorization methods.

Exercise 1. Factorize $x^3 + 2\,x^2 + 3\,x + 6$.

Exercise 2. Factorize $4\,x^4 + 4\,x^2 + 1$.

Note. The integration of $\frac{B\,x + C}{(x^2+1)^k}$ with $k > 1$ will not be in the exam.

  - Examples.

Example 1. Let $B, C, a, b \in \mathbb{R}$. Integrate

$$\int \frac{B\,x + C}{(x+a)^2 + b^2}\,\mathrm{d}x. \tag{4}$$

Note that there are more than one cases to discuss.

Solution. When $b \neq 0$, set $t = \frac{x+a}{b}$. This gives

$$x = b\,t - a, \qquad \mathrm{d}x = b\,\mathrm{d}t \tag{5}$$

which leads to

$$\begin{aligned} \int \frac{B\,x + C}{(x+a)^2 + b^2}\,\mathrm{d}x &= \int \frac{B\,(b\,t - a) + C}{b^2\,t^2 + b^2}\,b\,\mathrm{d}t \\ &= \frac{1}{b}\int \frac{b\,B\,t + (C - B\,a)}{t^2+1}\,\mathrm{d}t \\ &= \frac{B}{2}\ln(t^2+1) + \frac{C - B\,a}{b}\arctan t + C \\ &= \frac{B}{2}\ln[(x+a)^2 + b^2] + \frac{C - B\,a}{b}\arctan\frac{x+a}{b} + C. \end{aligned} \tag{6}$$

Exercise 3. What if $b=0$?

Example 2. Integrate the following.

a) $\int \frac{2x}{(x-1)(x-2)}\,\mathrm{d}x$;

b) $\int \frac{x^4-1}{x^3-4x}\,\mathrm{d}x$;

c) $\int \frac{x^3-6}{x^4+6x^2+8}\,\mathrm{d}x$.

Solution.

a) Since $\deg(2x)<\deg((x-1)(x-2))$ and the denominator is already factorized, we start from step 3:

$$\frac{2x}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}. \tag{7}$$

Multiply both sides by $(x-1)$:

$$\frac{2x}{x-2}=A+\frac{x-1}{x-2}B. \tag{8}$$

Let $x\to 1$:

$$-2=A. \tag{9}$$

Similarly $B=4$. Therefore

$$\int \frac{2x}{(x-1)(x-2)}\,\mathrm{d}x=-2\ln|x-1|+4\ln|x-2|. \tag{10}$$

b) As $\deg(x^4-1)\geqslant\deg(x^3-4x)$, we first calculate

$$(x^4-1)\div(x^3-4x)=x \text{ with remainder } 4x^2-1. \tag{11}$$

Now we factorize

$$x^3-4x=x(x^2-4)=x(x-2)(x+2). \tag{12}$$

Write

$$\frac{4x^2-1}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2} \tag{13}$$

we have

$$A=\frac{1}{4},\qquad B=\frac{15}{8},\qquad C=\frac{15}{8}. \tag{14}$$

Therefore

$$\begin{aligned}\int \frac{x^4-1}{x^3-4x}\,\mathrm{d}x &= \int x\,\mathrm{d}x+\int\left[\frac{1/4}{x}+\frac{15/8}{x-2}+\frac{15/8}{x+2}\right]\mathrm{d}x\\ &= \frac{x^2}{2}+\frac{1}{4}\ln|x|+\frac{15}{8}\ln|x^2-4|+C.\end{aligned} \tag{15}$$

c) We factorize

$$x^4+6x^2+8=(x^2+2)(x^2+4). \tag{16}$$

Thus write

$$\frac{x^3-6}{x^4+6x^2+8}=\frac{Bx+C}{x^2+2}+\frac{Dx+E}{x^2+4}. \tag{17}$$

This gives

$$\begin{aligned} x^3-6 &= (B\,x+C)\,(x^2+4)+(D\,x+E)\,(x^2+2) \\ &= (B+D)\,x^3+(C+E)\,x^2+(4\,B+2\,D)\,x+(4\,C+2\,E) \end{aligned} \tag{18}$$

and consequently

$$B+D = 1; \tag{19}$$
$$C+E = 0; \tag{20}$$
$$4\,B+2\,D = 0; \tag{21}$$
$$4\,C+2\,E = -6. \tag{22}$$

Solving this system we have

$$B=-1, \quad C=-3, \quad D=2, \quad E=3. \tag{23}$$

Thus

$$\begin{aligned} \int \frac{x^3-6}{x^4+6\,x^2+8}\,\mathrm{d}x &= \int \left[\frac{-x-3}{x^2+2}+\frac{2\,x+3}{x^2+4}\right]\mathrm{d}x \\ &= -\frac{1}{2}\ln(x^2+2)-\frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) \\ &\quad +\ln(x^2+4)+\frac{3}{2}\arctan\left(\frac{x}{2}\right)+C. \end{aligned} \tag{24}$$

Exercise 4. Calculate $\int \frac{3\,x-7}{x^3+x^2+4\,x+4}\,\mathrm{d}x$. (Ans:[1])

- Integration of $R(\sin x, \cos x)$.
  - Reduction to integration of rational functions through change of variable.
    - Universal change of variable:

      $$t=\tan\frac{x}{2}. \tag{25}$$

      This gives

      $$\cos x=\frac{1-t^2}{1+t^2}; \qquad \sin x=\frac{2\,t}{1+t^2}; \qquad \mathrm{d}x=\frac{2}{1+t^2}. \tag{26}$$

    - In many cases $t=\cos x$, $t=\sin x$ or $t=\tan x$ could be more efficient.

      Exercise 5. Prove that $\int R(\tan x)\,\mathrm{d}x$ can always be done through change of variable $t=\tan x$.

  - Examples.

    Example 3. Calculate the following.

    a) $\int \frac{\cos^3 x}{\sin^4 x}\,\mathrm{d}x$;

    b) $\int \tan^4 x\,\mathrm{d}x$;

    c) $\int \frac{\mathrm{d}x}{\cos^8 x}$;

1. $\ln\frac{x^2+4}{(x+1)^2}+\frac{1}{2}\arctan\frac{x}{2}+C$.

d) $\displaystyle\int \frac{\mathrm{d}x}{4-5\sin x}$.

Solution.

a) We have

$$\begin{aligned}\int \frac{\cos^3 x}{\sin^4 x}\,\mathrm{d}x &\overset{t=\sin x}{=\!=\!=} \int \frac{1-t^2}{t^4}\,\mathrm{d}t\\ &= \int [t^{-4}-t^{-2}]\,\mathrm{d}t\\ &= -\frac{1}{3\,t^3}+\frac{1}{t}+C\\ &= -\frac{1}{3\sin^3 x}+\frac{1}{\sin x}+C. \end{aligned} \tag{27}$$

Note. It is interesting that when I let wolframalpha.com calculate this integral, there was an intermediate step full of $\frac{x}{2}$ before $-\frac{1}{6}\,(3\,\cos(2\,x)-1)\,\csc^3 x+$ constant is shown. So looks like wolframalpha is calculating through the universal change of variable $t=\tan\frac{x}{2}$.

b) We have

$$\begin{aligned}\int \tan^4 x\,\mathrm{d}x &\overset{t=\tan x}{=\!=\!=} \int \frac{t^4}{t^2+1}\,\mathrm{d}t\\ &= \int \left[t^2-1+\frac{1}{t^2+1}\right]\mathrm{d}t\\ &= \frac{t^3}{3}-t+\arctan t+C\\ &= \frac{\tan^3 x}{3}-\tan x+x+C. \end{aligned} \tag{28}$$

c) We have

$$\begin{aligned}\int \frac{\mathrm{d}x}{\cos^8 x} &\overset{t=\tan x}{=\!=\!=} \int (t^2+1)^3\,\mathrm{d}t\\ &= \int [t^6+3\,t^4+3\,t^2+1]\,\mathrm{d}t\\ &= \frac{t^7}{7}+\frac{3}{5}\,t^5+t^3+t+C\\ &= \frac{1}{7}\tan^7 x+\frac{3}{5}\tan^5 x+\tan^3 x+\tan x+C. \end{aligned} \tag{29}$$

d) We have

$$\begin{aligned}\int \frac{\mathrm{d}x}{4-5\sin x} &\overset{t=\tan\frac{x}{2}}{=\!=\!=} \int \frac{\frac{2}{1+t^2}\,\mathrm{d}t}{4-5\,\frac{2\,t}{1+t^2}}\\ &= \int \frac{1}{2+2\,t^2-5\,t}\,\mathrm{d}t\\ &= \int \frac{\mathrm{d}t}{(2\,t-1)\,(t-2)}\\ &= \frac{1}{3}\ln\left|\frac{t-2}{2\,t-1}\right|+C\\ &= \frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}\right|+C. \end{aligned} \tag{30}$$