MATH 118 WINTER 2015 LECTURE 17 (FEB. 2, 2015)

Midterm 1 Review 1: Basics

- Things to remember:
  - Minimal table of indefinite integrals.

$$\int x^\alpha \, \mathrm{d}x = \frac{1}{1+\alpha} x^{1+\alpha} + C \qquad \alpha \in \mathbb{R}, \quad \alpha \neq 1; \tag{1}$$

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C; \tag{2}$$

$$\int e^x \, \mathrm{d}x = e^x + C; \tag{3}$$

$$\int \cos x \, \mathrm{d}x = \sin x + C; \tag{4}$$

$$\int \sin x \, \mathrm{d}x = -\cos x + C; \tag{5}$$

$$\int \frac{\mathrm{d}x}{(\cos x)^2} = \tan x + C;$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C; \tag{6}$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C. \tag{7}$$

  - Trig identities.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y; \tag{8}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y; \tag{9}$$

$$\sin(-x) = -\sin x; \tag{10}$$

$$\cos(-x) = -\cos x; \tag{11}$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x; \tag{12}$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x; \tag{13}$$

$$\sin(\pi - x) = \sin x. \tag{14}$$

$$\frac{1}{\cos^2 x} = \tan^2 x + 1. \tag{15}$$

Exercise 1. Write down formulas for $\sin(2\,x)$ and $\cos(2\,x)$.

Exercise 2. $\tan(x+y) = ?$ $\tan\left(\frac{\pi}{2} - x\right) = ?$

Exercise 3. $\sin(\pi + x) = ?$.

  - Binomial expansion.

$$(a+b)^2 = a^2 + 2\,a\,b + b^2; \tag{16}$$

$$(a+b)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3; \tag{17}$$

$$\vdots \quad \vdots \quad \vdots$$

Exercise 4. $(a+b)^4 = ?$.

  - Differential

$$\mathrm{d}f(x) := f'(x)\,\mathrm{d}x. \tag{18}$$

- Theorems (Practical versions).
  - FTC1:
    - f integrable on $[a,b]$;
    - $F' = f$ on (a,b);
    - F continuous on (a,b),

    then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)|_a^b. \tag{19}$$

  - Integration by substitution (change of variables).
    - Indefinite integral:
      - Type I substitution. $f(x) = f_1(u(x))\,u'(x)\,\mathrm{d}x \Longrightarrow$

$$\int f(x)\,\mathrm{d}x = \int f_1(u)\,\mathrm{d}u = F_1(u) + C = F_1(u(x)) + C; \tag{20}$$

      - Type II substitution. Set $x = x(t)$,

$$\int f(x)\,\mathrm{d}x = \int f(x(t))\,x'(t)\,\mathrm{d}t = G(t) + C = G(T(x)) + C. \tag{21}$$

        Here $T(x)$ is the inverse function to $x(t)$.
    - Definite integral: If
      - u, u' continuous on $[a,b]$, f continuous on $u([a,b])$, then

$$\int_a^b f(u(t))\,u'(t)\,\mathrm{d}t = \int_{u(a)}^{u(b)} f(x)\,\mathrm{d}x. \tag{22}$$

  - Integration by parts.
    - Indefinite integral: If u, v, u', v' are continuous then

$$\int u(x)\,v'(x)\,\mathrm{d}x = u(x)\,v(x) - \int v(x)\,u'(x)\,\mathrm{d}x. \tag{23}$$

    - Definite integral: If u, v, u', v' are continuous on $[a,b]$, then

$$\int_a^b u(x)\,v'(x)\,\mathrm{d}x = u(x)\,v(x)|_a^b - \int_a^b v(x)\,u'(x)\,\mathrm{d}x. \tag{24}$$

Exercise 5. Calculate $\int \frac{\mathrm{d}x}{(\sin x)^2}$ using change of variable $t = \frac{\pi}{2} - x$.

- Examples.

Example 1. Calculate the following indefinite integrals.

a) $\int \left(x^2 + \frac{1}{\sqrt[3]{x}}\right)^2 \mathrm{d}x$;

b) $\int \sqrt{x^2+2}\,x\,\mathrm{d}x$.

c) $\int \left(\tan 3\,x + \cot \frac{x}{3}\right) \mathrm{d}x$;

d) $\displaystyle\int \frac{\mathrm{d}x}{e^x + e^{-x}}$;

e) $\displaystyle\int \frac{\arcsin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$;

f) $\displaystyle\int \frac{\mathrm{d}x}{x + x^a}$; Here $a \in \mathbb{R}$.

Solution.

a) We have

$$\begin{aligned} \int \left(x^2 + \frac{1}{\sqrt[3]{x}}\right)^2 \mathrm{d}x &= \int (x^2 + x^{-1/3})^2\,\mathrm{d}x \\ &= \int \left(x^4 + 2\,x^{5/3} + x^{-2/3}\right)\mathrm{d}x \\ &= \frac{1}{5}\,x^5 + \frac{3}{4}\,x^{8/3} + 3\,x^{1/3} + C. \end{aligned} \tag{25}$$

b) We have

$$\begin{aligned} \int \sqrt{x^2 + 2}\,x\,\mathrm{d}x &= \frac{1}{2}\int (x^2 + 2)^{1/2}\,\mathrm{d}(x^2 + 2) \\ &= \frac{1}{3}\,(x^2 + 2)^{3/2} + C. \end{aligned} \tag{26}$$

c) We have

$$\begin{aligned} \int \left(\tan 3\,x + \cot\frac{x}{3}\right)\mathrm{d}x &= \int \frac{\sin 3\,x}{\cos 3\,x}\,\mathrm{d}x + \int \frac{\cos(x/3)}{\sin(x/3)}\,\mathrm{d}x \\ &= -\frac{1}{3}\int \frac{\mathrm{d}(\cos 3\,x)}{\cos 3\,x} + 3\int \frac{\mathrm{d}\left(\sin\frac{x}{3}\right)}{\sin\frac{x}{3}}\,\mathrm{d}x \\ &= -\frac{1}{3}\ln|\cos 3\,x| + 3\ln\left|\sin\frac{x}{3}\right| + C. \end{aligned} \tag{27}$$

d) We have

$$\begin{aligned} \int \frac{\mathrm{d}x}{e^x + e^{-x}} &= \int \frac{e^x\,\mathrm{d}x}{1 + (e^x)^2} \\ &= \int \frac{\mathrm{d}(e^x)}{1 + (e^x)^2} \\ &= \arctan(e^x) + C. \end{aligned} \tag{28}$$

e) We have

$$\begin{aligned} \int \frac{\arcsin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x &\overset{u=\sqrt{x}}{=\!=\!=} \int \frac{\arcsin u}{u}\,\mathrm{d}(u^2) \\ &= 2\int \arcsin u\,\mathrm{d}u \\ &= 2\left[u\arcsin u - \int \frac{u}{\sqrt{1-u^2}}\,\mathrm{d}u\right] \\ &= 2\,u\arcsin u + \int \frac{\mathrm{d}(1-u^2)}{\sqrt{1-u^2}} \\ &= 2\,u\arcsin u + 2\sqrt{1-u^2} + C \\ &= 2\sqrt{x}\arcsin\sqrt{x} + 2\sqrt{1-x} + C. \end{aligned} \tag{29}$$

f) There are two cases $a = 1$ and $a \neq 1$.

When $a = 1$ we have

$$\int \frac{\mathrm{d}x}{x + x^a} = \int \frac{\mathrm{d}x}{2\,x} = \frac{1}{2}\ln|x| + C. \tag{30}$$

When $a \neq 1$ we have

$$\begin{aligned} \int \frac{\mathrm{d}x}{x + x^a} &= \int \frac{x^{-a}\,\mathrm{d}x}{1 + x^{1-a}} \\ &= \frac{1}{1-a}\int \frac{\mathrm{d}(1 + x^{1-a})}{1 + x^{1-a}} \\ &= \frac{1}{1-a}\ln|1 + x^{1-a}| + C. \end{aligned} \tag{31}$$

Example 2. Calculate the following definite integrals.

a) $\int_1^e x\,(\ln x)^2\,\mathrm{d}x$;

b) $\int_0^{\pi/4} \cos^3 x \sin x\,\mathrm{d}x$.

Solution.

a) We have

$$\begin{aligned} \int_1^e x\,(\ln x)^2\,\mathrm{d}x &= \int_1^e (\ln x)^2\,\mathrm{d}\left(\frac{x^2}{2}\right) \\ &= \frac{x^2\,(\ln x)^2}{2}\Big|_1^e - \int_1^e \frac{x^2}{2}\,2\ln x\,\frac{1}{x}\,\mathrm{d}x \\ &= \frac{e^2}{2} - \int_1^e x\ln x\,\mathrm{d}x \\ &= \frac{e^2}{2} - \int_1^e \ln x\,\mathrm{d}\left(\frac{x^2}{2}\right) \\ &= \frac{e^2}{2} - \left[\frac{x^2\ln x}{2}\Big|_1^e - \int_1^e \frac{x^2}{2}\,\frac{1}{x}\,\mathrm{d}x\right] \\ &= \frac{e^2}{2} - \frac{e^2}{2} + \int_1^e \frac{x}{2}\,\mathrm{d}x \\ &= \frac{1}{4}\,[e^2 - 1]. \end{aligned} \tag{32}$$

b) We have

$$\begin{aligned} \int_0^{\pi/4} \cos^3 x \sin x\,\mathrm{d}x &\overset{u = \cos x}{=\!=\!=} -\int_1^{\sqrt{2}/2} u^3\,\mathrm{d}u \\ &= \frac{u^4}{4}\Big|_{\sqrt{2}/2}^1 = \frac{3}{16}. \end{aligned} \tag{33}$$

Exercise 6. Calcualte

$$\int_0^2 x^3\,e^{2x}\,\mathrm{d}x; \qquad \int_0^\infty e^{-x}(\sin x)^3\,\mathrm{d}x. \tag{34}$$