Math 118 Winter 2015 Lecture 16 (Jan. 30, 2015)

Puzzle. [1]

$$\lim_{x \to 0} \frac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}. \tag{1}$$

Note. This lecture is based on *Inside Interesting Integrals* by Paul J. Nahin, Springer 2015.

- Symmetry.

Example 1. Calculate $\int_0^\infty \frac{\ln x}{1+x^2} \mathrm{d}x$.

Solution. We have

$$\begin{aligned} I &= \int_0^\infty \frac{\ln x}{1+x^2} \mathrm{d}x \\ &\overset{t=1/x}{=\!=} \int_\infty^0 \frac{-\ln t}{1+\frac{1}{t^2}} \left(-\frac{1}{t^2}\right) \mathrm{d}t \\ &= \int_\infty^0 \frac{\ln t}{1+t^2} \mathrm{d}t \\ &= -\int_0^\infty \frac{\ln t}{1+t^2} \mathrm{d}t = -I. \end{aligned} \tag{2}$$

Therefore $I = 0$.

Exercise 1. Calculate $\int_0^\infty \frac{\ln x}{2+x^2} \mathrm{d}x$.

Example 2. Calculate $\int_{-1}^1 \frac{\cos x}{e^{1/x}+1} \mathrm{d}x$.

Solution. We have

$$\begin{aligned} I &= \int_{-1}^1 \frac{\cos x}{e^{1/x}+1} \mathrm{d}x \\ &\overset{t=-x}{=\!=} -\int_1^{-1} \frac{\cos t}{e^{-1/t}+1} \mathrm{d}x \\ &= \int_{-1}^1 \frac{\cos x}{e^{-1/x}+1} \mathrm{d}x. \end{aligned} \tag{3}$$

Now consider

$$2I = \int_{-1}^1 \cos x \left[\frac{1}{e^{1/x}+1} + \frac{1}{e^{-1/x}+1}\right] \mathrm{d}x = \int_{-1}^1 \cos x \, \mathrm{d}x = 2\sin 1. \tag{4}$$

Therefore $I = \sin 1$.

Exercise 2. Calculate $\int_{-1}^1 \frac{|x|}{e^{1/\sin x}+1} \mathrm{d}x$.

Problem 1. Prove $\int_0^4 \frac{\ln x}{\sqrt{4x - x^2}} \mathrm{d}x = 0$. (Hint:[2])

1. Taken from V. I. Arnold, *Huygens & Barrow, Newton & Hooke*, Birkhauser, 1990. Arnold claimed that the only modern mathematician he met that could solve this in a few minutes is Gerd Faltings.

Example 3. Calculate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, \mathrm{d}x$.

Solution. We have

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \, \mathrm{d}x \\ &\overset{t=\frac{\pi}{2}-x}{=\!=\!=} \int_{\pi/2}^{0} \frac{\sqrt{\cos t}}{\sqrt{\cos t} + \sqrt{\sin t}} \, (-\mathrm{d}t) \\ &= \int_0^{\pi/2} \frac{\sqrt{\cos t}}{\sqrt{\sin t} + \sqrt{\cos t}} \, \mathrm{d}t. \end{aligned} \tag{5}$$

Thus $2\,I = \frac{\pi}{2} \Longrightarrow I = \frac{\pi}{4}$.

Example 4. Calculate $\int_0^{\pi/2} \ln(\sin x) \, \mathrm{d}x$.

Solution. We have

$$\begin{aligned} I &= \int_0^{\pi/2} \ln(\sin x) \, \mathrm{d}x \\ &\overset{t=\frac{\pi}{2}-x}{=\!=\!=} \int_{\pi/2}^{0} \ln(\cos t) \, (-\mathrm{d}t) \\ &= \int_0^{\pi/2} \ln(\cos t) \, \mathrm{d}t. \end{aligned} \tag{6}$$

This gives

$$\begin{aligned} 2\,I &= \int_0^{\pi/2} [\ln(\sin x) + \ln(\cos x)] \, \mathrm{d}x \\ &= \int_0^{\pi/2} \ln(\sin 2\,x) \, \mathrm{d}x - \frac{\pi}{2} \ln 2 \\ &\overset{u=2x}{=\!=\!=} \frac{1}{2} \int_0^{\pi} \ln(\sin u) \, \mathrm{d}u - \frac{\pi}{2} \ln 2. \end{aligned} \tag{7}$$

Finally,

$$\begin{aligned} \int_0^{\pi} \ln(\sin u) \, \mathrm{d}u &= \int_0^{\pi/2} \ln(\sin u) \, \mathrm{d}u + \int_{\pi/2}^{\pi} \ln(\sin u) \, \mathrm{d}u \\ &= I + \int_{\pi/2}^{\pi} \ln(\sin(\pi - u)) \, \mathrm{d}u \\ &= I + I = 2\,I. \end{aligned} \tag{8}$$

Putting everything together we have

$$2\,I = I - \frac{\pi}{2} \ln 2 \Longrightarrow I = -\frac{\pi}{2} \ln 2. \tag{9}$$

Problem 2. Calculate $\int_0^1 \frac{\ln(1+x)}{1+x^2} \, \mathrm{d}x$. (Hint:[3])

2. Split 0 to 2 and 2 to 4.

3. $x = \tan t$. $u = \frac{\pi}{4} - t$.

- Infinite series.

Example 5. Calculate Bernoulli's integral $\int_0^1 x^x \, \mathrm{d}x$.

Solution. Recall $x^x = \exp(x \ln x)$. Thus apply Taylor expansion of exp we have

$$\begin{aligned}
\int_0^1 x^x \, \mathrm{d}x &= \int_0^1 e^{x \ln x} \, \mathrm{d}x \\
&= \int_0^1 \sum_{n=0}^\infty \frac{x^n (\ln x)^n}{n!} \, \mathrm{d}x \\
&= \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 x^n (\ln x)^n \, \mathrm{d}x \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^{n+1}}. \qquad (10)
\end{aligned}$$

Thus we finally have

$$\int_0^1 x^x \, \mathrm{d}x = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots \qquad (11)$$

Remark 6. In the above example, theoretical justifications are needed for the steps with $=$. We will discuss this issue more in a few weeks.

Exercise 3. Calculate $\int_0^1 x^n (\ln x)^n \, \mathrm{d}x$.

Exercise 4. Is the function x^x bounded on $[0, 1]$? What is 0^0?

Exercise 5. Prove the the infinite series in (11) converges.

Problem 3. Calculate $\int_0^1 x^{-x} \, \mathrm{d}x$.

Problem 4. Use $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ to calculate $\int_0^1 \frac{\ln(1+x)}{x} \, \mathrm{d}x$. (Ans:[4])

- Differentiating a parameter.

Example 7. Calculate $\int_0^\infty \frac{1}{(x^2+1)^2} \, \mathrm{d}x$.

Solution. Define

$$I(a) := \int_0^\infty \frac{1}{x^2 + a^2} \, \mathrm{d}x = \frac{\pi}{2\,a}. \qquad (12)$$

Therefore

$$-\frac{\pi}{2\,a^2} = \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\pi}{2\,a}\right) = I'(a) = -\int_0^\infty \frac{2\,a}{(x^2+a^2)^2} \, \mathrm{d}x. \qquad (13)$$

In particular,

$$-\frac{\pi}{2} = I'(1) = -\int_0^\infty \frac{2}{(x^2+1)^2} \, \mathrm{d}x \qquad (14)$$

and therefore

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4}. \qquad (15)$$

Remark 8. Again, the $=$ indicates steps that need theoretical justification.

4. $\pi^2/12$.

Example 9. Calculate $I := \int_0^\infty e^{-x^2/2} \, \mathrm{d}x$.

Solution. Define

$$g(t) := \left(\int_0^t e^{-x^2/2} \, \mathrm{d}x \right)^2. \tag{16}$$

Then

$$\begin{aligned} g'(t) &= 2 \int_0^t e^{-\frac{t^2+x^2}{2}} \, \mathrm{d}x \\ &\overset{y=x/t}{=\!=} \int_0^1 2\,t\,e^{-\frac{(1+y^2)t^2}{2}} \, \mathrm{d}y \\ &= \left(-2 \int_0^1 \frac{e^{-\frac{(1+y^2)t^2}{2}}}{1+y^2} \, \mathrm{d}y \right)'. \end{aligned} \tag{17}$$

Thus we have

$$g(t) = -2 \int_0^1 \frac{e^{-\frac{(1+y^2)t^2}{2}}}{1+y^2} \, \mathrm{d}y + C \tag{18}$$

for some constant C.

Taking limit $t \to \infty$ in (18) we have

$$I^2 = C. \tag{19}$$

Exercise 6. Prove $\lim_{t\to\infty} \int_0^1 \frac{e^{-(1+y^2)t^2/2}}{1+y^2} \, \mathrm{d}y = 0$.

Taking limit $t \to 0+$ in we have

$$0 = -2 \int_0^1 \frac{\mathrm{d}y}{1+y^2} + C \Longrightarrow C = \frac{\pi}{2}. \tag{20}$$

Thus $I = \sqrt{\frac{\pi}{2}}$.

Exercise 7. Calculate $\int_0^\infty \sqrt{-\ln x} \, \mathrm{d}x$. (Ans:[5])

Problem 5. Calculate $\int_{-\infty}^\infty e^{-x^2-bx} \, \mathrm{d}x$ for $b \in \mathbb{R}$. Then calculate $\int_{-\infty}^\infty x\,e^{-x^2-x} \, \mathrm{d}x$.

5. $\frac{\sqrt{\pi}}{2}$.