Math 118 Winter 2015 Homework 4 Solutions

Due Thursday Feb. 5 3pm in Assignment Box

Question 1. (5 pts) *Calculate the following integrals.*

a) (2 pts) $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,\mathrm{d}x.$

b) (3 pts) $\displaystyle\int \frac{\cos x}{\sin x + 2\cos x}\,\mathrm{d}x.$

Solution.

a) We have

$$\begin{aligned}
\int \frac{\sin 2x}{1+\cos^2 x}\,\mathrm{d}x &= 2\int \frac{\sin x \cos x}{1+\cos^2 x}\,\mathrm{d}x && (1)\\
&= -2\int \frac{\cos x}{1+\cos^2 x}\,\mathrm{d}\cos x && (2)\\
&\overset{u=\cos x}{=\!=} -2\int \frac{u}{1+u^2}\,\mathrm{d}u && (3)\\
&= -\ln(1+u^2)+C && (4)\\
&= -\ln(1+\cos^2 x)+C. && (5)
\end{aligned}$$

b) Setting $t=\tan\frac{x}{2}$, we have

$$\begin{aligned}
\int \frac{\cos x}{\sin x + 2\cos x}\,\mathrm{d}x &= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{2-2t^2}{1+t^2}}\,\frac{2}{1+t^2}\,\mathrm{d}t && (6)\\
&= \int \frac{t^2-1}{(t^2-t-1)(1+t^2)}\,\mathrm{d}t. && (7)
\end{aligned}$$

As

$$t^2-t-1=\left(t-\frac{1+\sqrt{5}}{2}\right)\left(t-\frac{1-\sqrt{5}}{2}\right) \qquad (8)$$

We write

$$\frac{t^2-1}{\left(t-\frac{1+\sqrt{5}}{2}\right)\left(t-\frac{1-\sqrt{5}}{2}\right)(t^2+1)} = \frac{A}{t-\frac{1+\sqrt{5}}{2}} + \frac{B}{t-\frac{1-\sqrt{5}}{2}} + \frac{Ct+D}{t^2+1}. \qquad (9)$$

Multiply both sides by $t-\frac{1+\sqrt{5}}{2}$ and set $t=\frac{1+\sqrt{5}}{2}$, we have $A=\frac{1}{5}$ (The calculation can be simplified a bit[1]); Similarly we have $B=\frac{1}{5}$. Thus (9) is simplified to

$$\frac{t^2-1}{(t^2-t-1)(t^2+1)} = \frac{1}{5}\frac{2t-1}{t^2-t-1} + \frac{Ct+D}{t^2+1}. \qquad (10)$$

Now setting $t=0$ we have

$$\frac{-1}{(-1)\,1} = \frac{1}{5}\frac{-1}{-1} + D \Longrightarrow D = \frac{4}{5}. \qquad (11)$$

1. To simplify the calculation, notice that for $t=\frac{1+\sqrt{5}}{2}$ we have $t^2-1=t$.

Finally multiply both sides by $(t^2-t-1)(t^2+1)$ and compare the coefficient for t^3 we see $C=-\frac{2}{5}$.

Thus

$$\int \frac{t^2-1}{(t^2-t-1)(1+t^2)}\,\mathrm{d}t=\frac{1}{5}\left[\ln(t^2-t-1)-\ln(t^2+1)\right]+\frac{4}{5}\arctan t+C. \tag{12}$$

Substitute back $t=\tan\frac{x}{2}$ we have

$$\begin{aligned}\int \frac{\cos x}{\sin x+2\cos x}\,\mathrm{d}x &= \frac{1}{5}\left[\ln\!\left(\left(\sin\frac{x}{2}\right)^2-\sin\frac{x}{2}\cos\frac{x}{2}-\left(\cos\frac{x}{2}\right)^2\right)+2\,x\right]+C\\ &= \frac{1}{5}\left[\ln(\sin x+2\cos x)+2\,x\right]+C.\end{aligned} \tag{13}$$

QUESTION 2. (5 PTS) *Calculate the following integrals.*

a) (2 PTS) $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{1+\sqrt[3]{x^2}}}$.

b) (3 PTS) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2\,(x-1)^4}}$.

Solution.

a) Let $t=\sqrt{1+\sqrt[3]{x^2}}$. We have $x^2=(t^2-1)^3$ and

$$\int \frac{x\,\mathrm{d}x}{\sqrt{1+\sqrt[3]{x^2}}}=\frac{1}{2}\int \frac{\mathrm{d}(x^2)}{t}=\frac{1}{2}\int \frac{6\,(t^2-1)^2\,t}{t}\,\mathrm{d}t=\frac{3}{5}\,t^5-2\,t^3+3\,t+C. \tag{14}$$

Substituting t back we have

$$\int \frac{x\,\mathrm{d}x}{\sqrt{1+\sqrt[3]{x^2}}}=\frac{3}{5}\,(1+x^{2/3})^{5/2}-2\,(1+x^{2/3})^{3/2}+3\,(1+x^{2/3})^{1/2}+C. \tag{15}$$

It can be simplified to

$$\frac{1}{5}\,(1+x^{2/3})^{1/2}\left[3\,x^{4/3}-4\,x^{2/3}+8\right]+C. \tag{16}$$

b) We write

$$\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2\,(x-1)^4}}=\int \frac{\mathrm{d}x}{(x^2-1)\left(\frac{x-1}{x+1}\right)^{1/3}}. \tag{17}$$

Now set $t=\left(\frac{x-1}{x+1}\right)^{1/3}$, we have $x=\frac{1+t^3}{1-t^3}$ and the integral is transformed to

$$\frac{3}{2}\int \frac{1}{t^2}\,\mathrm{d}t=-\frac{3}{2}\,\frac{1}{t}+C. \tag{18}$$

Therefore

$$\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2\,(x-1)^4}}=-\frac{3}{2}\left(\frac{x+1}{x-1}\right)^{1/3}+C. \tag{19}$$

QUESTION 3. (5 PTS) *Consider* $F_k(x):=\displaystyle\int \sqrt[3]{x^k+x^{-k}}\,\mathrm{d}x$ *for* $k=1,2,3$. *Apply Chebyshev's Theorem (Lecture 12) to determine which* $F_k(x)$ *is elementary and calculate these elementary ones.*

Solution. We have

$$\sqrt[3]{x^k + x^{-k}} = \left(\frac{1 + x^{2k}}{x^k}\right)^{1/3} = x^{-k/3} (1 + x^{2k})^{1/3}. \tag{20}$$

Thus we have $m = -\frac{k}{3}, n = 2k, p = \frac{1}{3}$. Clearly $p \notin \mathbb{Z}$. We have $\frac{m+1}{n} = \frac{1}{2k} - \frac{1}{6} \in \mathbb{Z}$ only when $k = 3$. Finally $\frac{m+1}{n} + p = \frac{1}{2k} + \frac{1}{6} \notin \mathbb{Z}$. Thus the only elementary function is $F_3(x)$.

To calculate $F_3(x)$ we set $t = (1 + x^6)^{1/3}$. Then $x = (t^3 - 1)^{1/6}$ and consequently

$$x^{-1} (1 + x^6)^{1/3} \, \mathrm{d}x = \frac{t^3}{2(t^3 - 1)} \, \mathrm{d}t \tag{21}$$

and we have

$$\begin{aligned}
\int \sqrt[3]{x^3 + x^{-3}} \, \mathrm{d}x &= \frac{1}{2} \int \frac{t^3}{t^3 - 1} \, \mathrm{d}t && (22)\\
&= \frac{t}{2} + \frac{1}{2} \int \frac{\mathrm{d}t}{t^3 - 1} && (23)\\
&= \frac{t}{2} - \frac{1}{12} \ln(t^2 + t + 1) + \frac{1}{6} \ln|t - 1| - \frac{\sqrt{3}}{6} \arctan\left(\frac{2t + 1}{\sqrt{3}}\right) + C && (24)\\
&= \frac{(1 + x^6)^{1/3}}{2} - \frac{1}{12} \ln\big((1 + x^6)^{2/3} + (1 + x^6)^{1/3} + 1\big) \\
&\quad + \frac{1}{6} \ln\big|(1 + x^6)^{1/3} - 1\big| - \frac{\sqrt{3}}{6} \arctan\left(\frac{2(1 + x^6)^{1/3} + 1}{\sqrt{3}}\right) + C. && (25)
\end{aligned}$$

Question 4. (5 pts) *Apply the results in Lecture 13 to prove that* $\int x \exp(x^3) \, \mathrm{d}x$ *is not elementary.*

Proof. Assume the contrary. Then there is a rational function $R(x)$ such that

$$\int x \exp(x^3) = R(x) \exp(x^3) + C. \tag{26}$$

Differentiating, we reach

$$x \exp(x^3) = R'(x) \exp(x^3) + 3x^2 R(x) \exp(x^3). \tag{27}$$

Cancelling the exponential we have

$$x = R'(x) + 3x^2 R(x). \tag{28}$$

As $R(x)$ is rational there are polynomials $P(x), Q(x)$ such that $R(x) = \frac{P(x)}{Q(x)}$. Substituting into (28) we have

$$x = \frac{P' Q - P Q'}{Q^2} + 3x^2 \frac{P}{Q} \Longrightarrow x Q^2 = P' Q - P Q' + 3x^2 P Q. \tag{29}$$

This leads to

$$P Q' = P' Q + 3x^2 P Q - x Q^2 = Q\,[P' + 3x^2 P - x Q]. \tag{30}$$

Let $(x - a)$ be a factor of Q with power $k \geqslant 1$ but not $k + 1$. Then we have $(x - a)^k$ dividing the RHS of (30) but not the LHS. Therefore Q has to be a constant. Wlog $Q(x) = 1$. Now (28) becomes

$$x = P'(x) + 3x^2 P(x) \tag{31}$$

where $P(x)$ is a polynomial. Let $\deg P = n \in \mathbb{N}$. Then the degree on the RHS is $n + 2$ while the degree on the LHS is 1. So $n + 2 = 1 \Longrightarrow n = -1 \notin \mathbb{N}$. Contradiction.

Therefore $\int x \exp(x^3) \, \mathrm{d}x$ is not elementary. $\square$