Math 118 Winter 2015 Lecture 14 (Jan. 28, 2015)

- Integration by parts for definite integrals.

THEOREM 1. (INTEGRATION BY PARTS) *If u, v are continuous on $[a, b]$ an differentiable on (a, b), and if u', v' are integrable on $[a, b]$, then*

$$\int_a^b u(x)\, v'(x)\, \mathrm{d}x = u(b)\, v(b) - u(a)\, v(a) - \int_a^b u'(x)\, v(x)\, \mathrm{d}x. \tag{1}$$

Proof. Let $F = u\, v$. Then $F' = u\, v' + u'\, v$. Since u, v are continuouson $[a, b]$, they are also integrable on $[a, b]$. Together with integrability of u', v' we conclude F' is integrable on $[a, b]$. Application of the first version of FTC gives the desired result. □

Problem 1. Prove the integration by parts formula using definition of Riemann integral only.

NOTATION. *It is often convenient to write $u(b)\, v(b) - u(a)\, v(a)$ as $u\, v|_a^b$.*

Example 2. Calculate

$$\int_1^2 x \ln x\, \mathrm{d}x. \tag{2}$$

Solution. We have

$$\begin{aligned}\int_1^2 x \ln x\, \mathrm{d}x &= \frac{1}{2} \int_1^2 \ln x\, \mathrm{d}x^2 \\ &= \frac{1}{2} \left[x^2 \ln x|_1^2 - \int_1^2 x\, \mathrm{d}x \right] \\ &= 2 \ln 2 - \frac{3}{4}. \end{aligned} \tag{3}$$

Exercise 1. Calculate $\int_1^2 x^3 (\ln x)^2\, \mathrm{d}x$.

- Change of variables for definite integrals.

THEOREM 3. *Let u be continuous on $[a, b]$, differentiable on (a, b) and assume u' is integrable on $[a, b]$. If f is continuous on $I := u([a, b])$, then*

$$\int_a^b f(u(t))\, u'(t)\, \mathrm{d}t = \int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x. \tag{4}$$

Remark 4. Recall that in the case $u(b) < u(a)$, the integral is understood as

$$\int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x = -\int_{u(b)}^{u(a)} f(x)\, \mathrm{d}x. \tag{5}$$

Proof. We notice that, if we define $F(x) = \int_{u(a)}^{x} f(t)\, \mathrm{d}t$, then $F'(x) = f(x)$ and it follows from FTC Version 1 that

$$\int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x = F(u(b)) - F(u(a)); \tag{6}$$

On the other hand, if we set

$$G(t) := F(u(t)) \tag{7}$$

then by Chain rule

$$G'(t) = \frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = F'(u(t))\, u'(t) = f(u(t))\, u'(t). \tag{8}$$

Note that the last equality is a result of FTC Version 2 and only holds because f is continuous at every $u(t)$.

Next we check that $f(u(t))\ \ u'(t)$ is integrable: $f(x)$, $u(t)$ continuous $\Longrightarrow f(u(t))$ continuous $\Longrightarrow f(u(t))$ integrable $\Longrightarrow f(u(t))\, u'(t)$ integrable since $u'(t)$ is integrable.

Finally applying FTC Version 1 to G we have

$$\int_a^b f(u(t))\, u'(t)\, \mathrm{d}t = G(b) - G(a) = F(u(b)) - F(u(a)). \tag{9}$$

and the proof ends. Note that in this last step we need G to be continuous, which follows from the continuity of f and of u. □

Remark 5. Note that we **don't** need u to be one-to-one![1] In particular, it may happen that $u([a, b]) \neq [u(a), u(b)]$.

THEOREM 6. *Let $u(t)$: $[a, b] \mapsto \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and assume u' is continuous on $[a, b]$. Let $f(x)$ be integrable on $I := u([a, b])$. Further assume that u is strictly increasing or decreasing. Then*

$$\int_a^b f(u(t))\, u'(t)\, \mathrm{d}t = \int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x. \tag{10}$$

Proof. Wlog assume u is strictly increasing. Then $u([a, b]) = [u(a), u(b)]$. Further wlog assume that $a = 0, b = 1$. Let $P_n = \left\{0, \frac{1}{n}, \ldots, 1\right\}$. Denote by Q_n the corresponding partition of $[u(0), u(1)]$: $\left\{u(0), u\left(\frac{1}{n}\right), \ldots, u(1)\right\}$.

Exercise 2. Why is Q_n a partition?

Denote by $I_n := \sum_{k=1}^n f(u(c_{n,k})) \left(u\left(\frac{k}{n}\right) - u\left(\frac{k-1}{n}\right)\right)$ where $c_{n,k} \in \left(\frac{k-1}{n}, \frac{k}{n}\right)$ comes from MVT:

$$u'(c_{n,k}) = n \left(u\left(\frac{k}{n}\right) - u\left(\frac{k-1}{n}\right)\right). \tag{11}$$

. As f is integrable on $[u(0), u(1)]$ and u is continuous on $[0, 1]$, we have

$$\lim_{n \to \infty} I_n = \int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x \tag{12}$$

Exercise 3. Prove (12).

By our choices of $c_{n,k}$ there holds

$$I_n = \sum_{k=1}^n f(u(c_{n,k}))\, u'(c_{n,k}) \left(\frac{k}{n} - \frac{k-1}{n}\right). \tag{13}$$

1. In higher dimensions, we do need the change of variable function to be one-to-one. To fully understand this issue, check out "degree theory".

Now we have

$$\begin{aligned}
|U(f(u)\,u', P_n) - I_n| &= \frac{1}{n}\left|\sum_{k=1}^{n}\left[\sup_{t\in\left[\frac{k-1}{n},\frac{k}{n}\right]} f(u(t))\,u'(t) - f(u(c_{n,k}))\,u'(c_{n,k})\right]\right| \\
&= \frac{1}{n}\left|\sum_{k=1}^{n}\sup_{t\in\left[\frac{k-1}{n},\frac{k}{n}\right]}[f(u(t))\,u'(t) - f(u(c_{n,k}))\,u'(c_k)]\right| \\
&\leqslant \frac{1}{n}\left|\sum_{k=1}^{n}\left(\sup_{t\in\left[\frac{k-1}{n},\frac{k}{n}\right]} f(u(t)) - f(u(c_{n,k}))\right)u'(c_k)\right| \\
&\quad + \frac{1}{n}\left|\sum_{k=1}^{n}\left(\sup_{t\in\left[\frac{k-1}{n},\frac{k}{n}\right]}|f(u(t))|\right)\sup_{t\in\left[\frac{k-1}{n},\frac{k}{n}\right]}|u'(t) - u'(c_k)|\right|. \qquad (14)
\end{aligned}$$

Exercise 4. Explain (14) and finish the proof. □

Remark 7. Checking (13) we see that $\lim_{n\to\infty} I_n = \int_a^b f(u(t))\,u'(t)\,\mathrm{d}t$ and the proof ends as long as $f(u(t))\;u'(t)$ is integrable on $[a,\,b]$. However it is not clear to me yet whether integrability of f and differentiability of u (without continuity of u') could guarantee this. Also it may happen that the monotonicity assumption could be dropped.

- Taylor expansion with integral remainder.

Example 8. Taylor expansion with integral remainder.

We can obtain Taylor expansion using integration by parts.

$$\begin{aligned}
f(x) - f(a) &= \int_a^x f'(t)\,\mathrm{d}t \\
&= -\int_a^x f'(t)\,\mathrm{d}(x-t) \\
&= -f'(t)\,(x-t)|_a^x + \int_a^x (x-t)\,\mathrm{d}f'(t) \\
&= f'(a)\,(x-a) + \int_a^x (x-t)\,f''(t)\,\mathrm{d}t \\
&= f'(a)\,(x-a) - \frac{1}{2}\int_a^x f''(t)\,\mathrm{d}(x-t)^2 \\
&= f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2 + \frac{1}{2}\int_a^x (x-t)^2\,f'''(t)\,\mathrm{d}t. \qquad (15)
\end{aligned}$$

Exercise 5. Prove

$$f(x) = \sum_{m=0}^{n}\frac{f^{(m)}(a)}{m!}(x-a)^m + \int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t. \qquad (16)$$

The remainder $\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t$ is called "integral form of the remainder" for the Taylor expansion of f. One can show that if $f^{(n+1)}(t)$ is continuous, then there is $c \in (a, x)$ such that

$$\int_a^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad (17)$$

which is exactly the Lagrange remainder.

Exercise 6. Prove (17).

The disadvantage of the Lagrange remainder is that

1. We have no knowledge of where c exactly is;
2. The dependence of c on x may be rough. For example, we can differentiate the integral remainder but not the Lagrange remainder (due to $c(x)$ may not be differentiable).

On the other hand, there is no problem calculating

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t\right]. \tag{18}$$

Exercise 7. Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t\right]$.

Therefore in analysis it is usually advantageous to the integral form for the remainder.