Math 118 Winter 2015 Lecture 13 (Jan. 26, 2015)

Note. All the functions below should be understood as complex functions of complex variables. However most of the main ideas could be roughly understood with this fact ignored.

Note. This lecture is based mainly on the following references.

- Joseph Fels Ritt, *Integration in Finite Terms: Liouville's Theory of Elementary Methods*, Columbia University Press, 1948.
- Elena Anne Marchisotto & Gholam-Ali Zakeri, *An Invitation to Integration in Finite Terms*, The College Mathematics Journal 25(4) September 1994 295 – 308.

- Chebyshev's Theorem.
  - Recall:

    Theorem 1. (Chebyshev 1853) *Let $r, s \in \mathbb{Q}$. Then $\int x^r (1-x)^s \, \mathrm{d}x$ is elementary if and only if one or more of the following holds: $r \in \mathbb{Z}, s \in \mathbb{Z}, r+s \in \mathbb{Z}$.*

  - We have proved the "if" part in our last lecture.
  - The "only if" part.
    - We won't be able to present the full proof here, instead we just try to present the main ideas to make the theorem sound reasonable.

      We prove by contradiction. Assume the integral is elementary.
    - Abel's theorem.
      - Algebraic function: A function u of x is algebraic if it is defined by an irreducible relation

        $$a_n(x)\, u^n + \cdots + a_1(x)\, u + a_0(x) = 0 \tag{1}$$

        where each $a_i(x)$ is a polynomial in x.

        Exercise 1. Show that the function $y(x) := x^r (1-x)^s$ is algebraic when r, $s \in \mathbb{Q}$.

        Exercise 2. Prove: All rational functions are algebraic.

      - Weak Liouville theorem.

        Theorem 2. (Liouville 1833) *Let $f(x)$ be algebraic and assume $\int f(x) \, \mathrm{d}x$ is elementary. Then*

        $$\int f(x) \, \mathrm{d}x = u_0(x) + c_1 \ln u_1(x) + \cdots + c_r \ln u_r(x) + C \tag{2}$$

        where $u_0(x), \ldots, u_r(x)$ are algebraic.

        The rigorous proof of this is somewhat tedious. On the other hand, it is easy to convince oneself of the conclusion: If there are exponentials or nonlinear involvement of logarithms, then they will survive differentiation and will appear in $f(x)$.[1]

1. Of course the fact that exponentials and logarithms are not algebraic function still needs to be proved.

- Abel's theorem.[2]

  THEOREM 3. (ABEL 1820S?) *The functions u_0, ..., u_r in (2) can be taken to be bi-rational in x, $f(x)$.*

  Exercise 3. Why is this an improvement on Theorem 2?

  There are two ways to prove this, through complex analysis or through algebra. We do not present them here.[3] On the other hand, Theorem 3 is intuitively correct as roots also "almost survives" differentiation, therefore any non-rational algebraic functions will leave its mark in the original integrand $f(x)$.

  – When the integral is elementary but not algebraic.

    In this case we have at least one $c_i \neq 0$. Taking derivative we obtain $\frac{u_i'}{u_i}$. Let $a \in \mathbb{C}$ be such that $u_i(a) = 0$. Then if we write everything in $x - a$, intuitively there will be a term $\frac{b}{x-a}$, as we can easily check with say $u = x^2, u = x^{-1}$, etc. We say the "residue" of u_i'/u_i is nonzero at a.

    On the other hand, there is no such term in u_0', as can be illustruated through examples like x^2 and x^{-1}. Therefore the residue of u_0' at any a is zero.

    PROPOSITION 4. *If $f(x)$ is an algebraic function whose integral is elementary but not algebraic, then there is a on a Riemann surface of f at which f has residue dintinct from zero.*

  – Residue analysis for $x^r (1-x)^s$.

    When $r \notin \mathbb{Z}$, the residue at 0 is zero; When $s \notin \mathbb{Z}$, the residue at 1 is zero. When $r + s \notin \mathbb{Z}$, the residue at ∞ is zero. The residue at any other point is obviously zero as $x^r (1-x)^s$ enjoys a Taylor expansion there and no Taylor expansion contains terms like $\frac{b}{x-a}$.

    Therefore if $\int x^r (1-x)^s \, \mathrm{d}x$ is elementary, it must be algebraic, and by Theorem 3 it must be a rational function of x and $y(x) := x^r (1-x)^s$.

  – Completing the proof of Chebyshev's theorem.

2. According to Ritt, Abel actually proved this theorem about ten years before Liouville's result.

3. The complex analysis proof does not make sense in real variables. On the other hand, the algebra proof can be sketched as follows. We consider the case where only u_0, u_1 presents. Rename them u, v. As u, v are algebraic they satisfy

$$a_m u^m + \cdots + a_0 = 0, \qquad b_n v^n + \cdots + b_0 = 0. \tag{3}$$

Let the roots be $u_1 = u$, ..., u_m, $v_1 = v$, ..., v_n. Now let $t{:} = h\, u + k\, v$ for some appropriate h, $k \in \mathbb{C}$. Define $F(z) := \prod_{i,j} (z - (h\, u_i + k\, v_j))$. Note that the coefficients of F are symmetric combinations of u_i, v_j and thus are rational (To understand why see e.g. pp. 39 - 40 of Emil Artin, *Galois Theory*, Dover 1998). Further define G_{ij}, H by $F = [z - (h\, u_i + k\, v_j)]\, G_{ij}(z)$, $H(z) = \sum_{i,j} G_{ij} u_i$. Then we have

$$H(t) = F'(t)\, u \Longrightarrow u \text{ is rational in } x, t \Longrightarrow \int f = u + c \ln v = U(t,x) + c \ln V(t,x) \tag{4}$$

where U, V are rationa.

Now t, a linear combination of u, v, is also algebraic over the field $\mathcal{F}(x, f)$ generated by rational functions of x together with f. Let $\alpha_l\, t^l + \cdots + \alpha_0 = 0$ (assume the LHS polynomial is irreducible in $\mathcal{F}(x, f)$) and let $t_1 = t, ..., t_l$ be the roots. From this it is clear that t' is rational in x, t and consequently $f(x)$ is rational in t (you need a bit knowledge of partial derivatives here) and $f = \frac{\mathrm{d}}{\mathrm{d}x}[U(t,x) + c \ln V(t,x)]$ becomes a equation with coefficients in $\mathcal{F}(x, f)$ and thus must have as a factor $\alpha_l\, t^l + \cdots + \alpha_0$. Consequently $\int f = U(t_i, x) + c \ln V(t_i, x)$ for every $i = 1, 2, ..., l$. Add them up and use symmetry, we reach our conclusion.

We have seen that if $\int x^r\,(1-x)^s\,\mathrm{d}x$ is elementary, then there is a (bi)rational function $R(x,y)$ such that

$$\int x^r\,(1-x)^s = R(x, x^r\,(1-x)^s). \tag{5}$$

Now assume $r=\frac{p}{m}, s=\frac{q}{m}$ where $p,q\in\mathbb{Z}, m\in\mathbb{N}$ and m is the smallest possible. Clearly we have

$$y^m(x) = x^p\,(1-x)^q. \tag{6}$$

Consequently, we can write

$$u(x) = \frac{P(x,y)}{Q(x,y)} \tag{7}$$

where the highest power of y in P,Q are no more than $m-1$. Furthermore we can find a $R(x,y)$ that is a polynomial in y with coefficients rational functions of x, such that $R(x,y)\,Q(x,y)=1$.

Example 5. We convince ourselves through an example for numbers. Let x be defined as $x^2=2$. We claim there is a linear polynomial $a\,x+b$ such that $(a\,x+b)\,(x+1)=1$. Expanding the left hand side we have

$$a\,x^2+(a+b)\,x+b=1. \tag{8}$$

Recalling $x^2=2$ we reach

$$(a+b)\,x+b=1-2\,a \tag{9}$$

and a,b can be determined through $a+b=0, b=1-2\,a$.

Therefore we can assume

$$u(x) = A_0(x)+A_1(x)\,y+\cdots+A_{m-1}(x)\,y^{m-1} \tag{10}$$

where each $A_i(x)$ is rational.

Exercise 4. Prove that

$$u'(x) = B_0(x)+B_1(x)\,y+\cdots+B_{m-1}(x)\,y^{m-1} \tag{11}$$

where each $B_i(x)$ is rational. (Hint:[4])

Thus we arrive at

$$y = B_0(x)+B_1(x)\,y+\cdots+B_{m-1}(x)\,y^{m-1} \tag{12}$$

(12) must be an identity as otherwise y satisfies an equation of degree less than m which is not possible. Therefore we have

$$u(x) = A(x)\,y(x). \tag{13}$$

Now we initiate the final step. Let $A(x)=\frac{P(x)}{Q(x)}$. Clearly the factors of Q can only be 0, 1. But then $u(x)$ is "more singular" at $0,1$ than its derivative y which is not possible. Therefore $A(x)$ is a polynomial. But then clearly we should have A linear as otherwise $u'(x)=y(x)$ will be "more singular" at ∞ than $y(x)$, a nonsensical statement.

4. $\frac{y'}{y}=\frac{1}{m\,y^m}\,(y^m)'$.

Thus we have to have

$$[(a+b\,x)\,x^r\,(1-x)^s]' = x^r\,(1-x)^s \tag{14}$$

which leads to

$$b\,x\,(1-x) + r\,(a+b\,x)\,(1-x) + s\,(a+b\,x)\,x = x\,(1-x) \tag{15}$$

and consequently

$$x|\,(a+b\,x), \qquad (1-x)|\,(a+b\,x) \tag{16}$$

from which it follows $a=b=0$. Contradiction.

- Liouville's Theorem and its applications.
  - Strong Liouville theorem.

THEOREM 6. (LIOUVILLE 1835)

a) *Let $F(x, y_1, \ldots, y_m)$ be algebraic and assume each $y_i'(x)$ is an algebraic function of $x, y_1, \ldots, y_m$. Then $\int F(x, y_1(x), \ldots, y_m(x))\,\mathrm{d}x$ is elementary if and only if*

$$\int F(x, y_1(x), \ldots, y_m(x))\,\mathrm{d}x = u_0(x) + \sum_{j=1}^{n} c_j \ln u_j(x) + C \tag{17}$$

where each $u_0, \ldots, u_n$ are algebraic functions of $x, y_1, \ldots, y_m$.

b) *Let $F(x, y_1, \ldots, y_m)$ be rational and each $y_i'(x)$ is rational in $x, y_1, \ldots, y_m$. Then the $u_0, \ldots, u_n$ in (17) are rational in $x, y_1, \ldots, y_m$.*

Example 7. Let F be rational. Then $F(x, e^x, \ln x, e^{e^x}, \ln(\ln x), \sin x, \cos x, \cos(e^x))$ satisfies the conditions in a) but not those in b).

COROLLARY 8. *Let $g(x), h(x)$ be rational. Then $\int h(x)\,e^{g(x)}\,\mathrm{d}x$ is elementary if and only if there is a rational function $R(x)$ such that*

$$\int h(x)\,e^{g(x)}\,\mathrm{d}x = R(x)\,e^{g(x)} + C. \tag{18}$$

Proof. We sketch it here.[5]

□

5. Denote $y := e^{g(x)}$ and u the integral. Then we have

$$u = U_0(x, y) + \sum c_i \ln U_i(x, y) \tag{19}$$

where $U_0, \ldots$ are rational. Taking derivative we have (you need to spend a couple of minutes/hours getting to know multi-variable chain rule here)

$$y\,h(x) = \frac{\partial U_0}{\partial y}\,y\,g' + \frac{\partial U_0}{\partial x} + \sum \frac{1}{U_i}\left[\frac{\partial U_i}{\partial y}\,y\,g' + \frac{\partial U_i}{\partial x}\right] \tag{20}$$

As y is not algebraic, (20) must be an identity in y which means we can safely replace every y by $t\,y$ for $t \in \mathbb{R}$. Integrate and then differentiate w.r.t. t and then set $t=1$, we conclude $u = U_0(x, y)$. Apply the same trick again we reach

$$y\,\frac{\partial U_0}{\partial y} = U_0 + C \tag{21}$$

for some constant C. This is also an identity so we can treat y as a variable and differentiate w.r.t. y to obtain $y\,\frac{\partial^2 U_0}{\partial y^2} = 0$ which means U_0 must be linear in y and the conclusion follows.

Example 9. Consider $\int e^{-x^2}\,\mathrm{d}x$. By Corollary 8 we see that if it is elementary, then there is a rational function $R(x)$ such that

$$\left[R(x)\,e^{-x^2}\right]' = e^{-x^2}. \tag{22}$$

This gives

$$R'(x) - 2\,x\,R(x) = 1. \tag{23}$$

Now let $R(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials. Then we have

$$P'(x)\,Q(x) - Q'(x)\,P(x) - 2\,x\,P(x)\,Q(x) = Q^2(x) \tag{24}$$

which gives

$$Q'(x)\,P(x) = Q(x)\,[P'(x) - 2\,x\,P(x) - Q(x)]. \tag{25}$$

Let $(x-a)^k|\,Q$ but $(x-a)^{k+1}\nmid Q$. Then we have $(x-a)^k|$ RHS but $(x-a)^k\nmid$LHS. Thus $k=0$. Wlog $Q=1$. Then

$$P'(x) - 2\,x\,P(x) = 1 \tag{26}$$

for some polynomial $P(x)$.

Exercise 5. Prove that there is no polynomial $P(x)$ satisfying (26).

Thus $\int e^{-x^2}\,\mathrm{d}x$ is not elementary.

Exercise 6. Prove that the following integrals are not elementary. Note that Corollary 8 may not directly apply to one or more of the following.

$$\int x^4\,e^{x^2}, \qquad \int \frac{e^x}{x}\,\mathrm{d}x, \qquad \int e^{x^3}\,\mathrm{d}x, \qquad \int \sqrt{x}\,e^x\,\mathrm{d}x, \qquad \int \frac{\mathrm{d}x}{\ln x}. \tag{27}$$