Math 118 Winter 2015 Homework 3 Solutions

Due Thursday Jan. 29 3pm in Assignment Box

Question 1. (15 pts) *Calculate the following indefinite integrals through partial fractions. Please provide enough details.*

a) (2 pts) $\displaystyle\int \frac{2\,x}{x^2+2\,x+2}\,\mathrm{d}x;$

b) (2 pts) $\displaystyle\int \frac{x^2+2}{(x+1)^3\,(x-2)}\,\mathrm{d}x;$

c) (2 pts) $\displaystyle\int \frac{2\,x\,\mathrm{d}x}{(x^2+1)\,(x-1)}.$

d) (3 pts) $\displaystyle\int \frac{x^4+4\,x^3+11\,x^2+12\,x+8}{(x^2+2\,x+2)^2\,(x+1)}\,\mathrm{d}x.$

e) (3 pts) $\displaystyle\int \frac{x^4}{x^4+x^3-x^2+x-2}\,\mathrm{d}x.$

f) (3 pts) $\displaystyle\int \frac{\mathrm{d}x}{x^6-1}.$

Solution.

a) We have

$$\int \frac{2\,x\,\mathrm{d}x}{x^2+2\,x+2} = \int \frac{(2\,x+2)\,\mathrm{d}x}{x^2+2\,x+2} - 2\int \frac{\mathrm{d}x}{x^2+2\,x+2} \tag{1}$$

$$= \int \frac{\mathrm{d}(x^2+2\,x+2)}{x^2+2\,x+2} - 2\int \frac{\mathrm{d}(x+1)}{(x+1)^2+1} \tag{2}$$

$$= \ln|x^2+2\,x+2| - 2\arctan(x+1) + C. \tag{3}$$

b) We write

$$\frac{x^2+2}{(x+1)^3\,(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-2}. \tag{4}$$

Multiply both sides by $x-2$ and set $x=2$ we have $D=\frac{2}{9}$. Multiply both sides by $(x+1)^3$ and set $x=-1$ we have $C=-1$. To decide A, B, we multiply both sides by $(x+1)^3\,(x-2)$:

$$x^2+2 = A\,(x+1)^2\,(x-2) + B\,(x+1)\,(x-2) + C\,(x-2) + D\,(x+1)^3. \tag{5}$$

Equating the coefficients for x^3 we have $0 = A + D \Longrightarrow A = -\frac{2}{9}$. Finally equating the constant term we have $2 = -2\,A - 2\,B - 2\,C + D \Longrightarrow B = \frac{1}{3}$.

Thus we have

$$\int \frac{x^2+2}{(x+1)^3\,(x-2)}\,\mathrm{d}x = \int \left[-\frac{2}{9}\,\frac{1}{x+1} + \frac{1}{3\,(x+1)^2} - \frac{1}{(x+1)^3} + \frac{2}{9\,(x-2)}\right]\mathrm{d}x$$

$$= -\frac{2\,x-1}{6\,(x+1)^2} + \frac{2}{9}\ln\left|\frac{x-2}{x+1}\right| + C. \tag{6}$$

c) We write

$$\frac{2\,x}{(x^2+1)\,(x-1)} = \frac{A\,x+B}{x^2+1} + \frac{C}{x-1}. \tag{7}$$

Multiply both sides by $x-1$ and set $x=1$ we have $C=1$. Set $x=0$ we have $0 = B - C \Longrightarrow B=1$. Finally comparing the x^2 term in

$$2\,x = (A\,x+B)\,(x-1) + C\,(x^2+1) \tag{8}$$

we obtain $0=A+C \Longrightarrow A=-1$.

Thus we have

$$\begin{aligned}\int \frac{2\,x\,\mathrm{d}x}{(x^2+1)\,(x-1)} &= \int \frac{-x}{x^2+1}\,\mathrm{d}x+\int \frac{\mathrm{d}x}{x^2+1}+\int \frac{\mathrm{d}x}{x-1}\\ &= -\frac{1}{2}\ln|x^2+1|+\arctan x+\ln|x-1|+C. \qquad (9)\end{aligned}$$

d) We write

$$\frac{x^4+4\,x^3+11\,x^2+12\,x+8}{(x^2+2\,x+2)^2\,(x+1)}=\frac{A\,x+B}{x^2+2\,x+2}+\frac{C\,x+D}{(x^2+2\,x+2)^2}+\frac{E}{x+1}. \qquad (10)$$

Multiply both sides by $x+1$ and set $x=-1$ we have $E=4$. To determine A—D, we multiply both sides by $(x^2+2\,x+2)^2\,(x+1)$ to obtain

$$x^4+4\,x^3+11\,x^2+12\,x+8=(A\,x+B)\,(x^2+2\,x+2)\,(x+1)+(C\,x+D)\,(x+1)+4\,(x^2+2\,x+2)^2. \qquad (11)$$

Comparing the x^4 terms we have $1=A+4 \Longrightarrow A=-3$. Comparing the $1, x, x^3$ terms[1] we obtain the following system for B, C, D:

$$\begin{aligned} 8 &= 2\,B+D+16; & (12)\\ 12 &= -6+4\,B+C+D+32; & (13)\\ 4 &= -9+B+16. & (14)\end{aligned}$$

From (14) we obtain $B=-3$. Then using (12) we have $D=-2$. Finally substituting $B=-3$, $D=-2$ into (13) we obtain $C=0$.

Thus we have

$$\frac{x^4+4\,x^3+11\,x^2+12\,x+8}{(x^2+2\,x+2)^2\,(x+1)} = -\frac{3\,(x+1)}{x^2+2\,x+2}-\frac{2}{(x^2+2\,x+2)^2}+\frac{4}{x+1}. \qquad (15)$$

We evaluate them one by one.

$$\begin{aligned}-3\int \frac{x+1}{x^2+2\,x+2}\,\mathrm{d}x &= -\frac{3}{2}\int \frac{2\,x+2}{x^2+2\,x+2}\,\mathrm{d}x & (16)\\ &= -\frac{3}{2}\int \frac{\mathrm{d}(x^2+2\,x+2)}{x^2+2\,x+2} & (17)\\ &= -\frac{3}{2}\ln(x^2+2\,x+2)+C. & (18)\end{aligned}$$

For the second integral we use the following trick:

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2+2\,x+2} &= \frac{x}{x^2+2\,x+2}-\int x\,\mathrm{d}\left(\frac{1}{x^2+2\,x+2}\right) & (19)\\ &= \frac{x}{x^2+2\,x+2}+\int \frac{2\,x^2+2\,x}{(x^2+2\,x+2)^2}\,\mathrm{d}x & (20)\\ &= \frac{x}{x^2+2\,x+2}+2\int \frac{\mathrm{d}x}{x^2+2\,x+2}-\int \frac{2\,x+4}{(x^2+2\,x+2)^2}\,\mathrm{d}x & (21)\\ &= \frac{x}{x^2+2\,x+2}+2\arctan(x+1)-\int \frac{2\,x+2}{(x^2+2\,x+2)^2}\,\mathrm{d}x\\ &\quad -2\int \frac{\mathrm{d}x}{(x^2+2\,x+2)^2} & (22)\\ &= \frac{x+1}{x^2+2\,x+2}+2\arctan(x+1)-2\int \frac{\mathrm{d}x}{(x^2+2\,x+2)^2}. & (23)\end{aligned}$$

1. Note that we choose x^3 so that C, D would not appear in this equation.

Thus we have

$$\int \frac{\mathrm{d}x}{(x^2+2\,x+2)^2} = \frac{1}{2}\left[\frac{x+1}{x^2+2\,x+2} + \arctan(x+1)\right] + C. \tag{24}$$

Now finally we can calculate

$$\begin{aligned}\int \frac{x^4+4\,x^3+11\,x^2+12\,x+8}{(x^2+2\,x+2)^2\,(x+1)}\,\mathrm{d}x &= -\frac{3}{2}\ln(x^2+2\,x+2)+4\ln|x+1| \\ &\quad -\frac{x+1}{x^2+2\,x+2} - \arctan(x+1) + C. \end{aligned}\tag{25}$$

e) First we notice that $\deg x^4 = \deg\,(x^4+x^3-x^2+x-2)$. Therefore we have to perform polynomial division to obtain

$$x^4 = 1\cdot(x^4+x^3-x^2+x-2) + (-x^3+x^2-x+2). \tag{26}$$

Therefore

$$\frac{x^4}{x^4+x^3-x^2+x-2} = 1 + \frac{-x^3+x^2-x+2}{x^4+x^3-x^2+x-2}. \tag{27}$$

Next we factorize the denominator. We see that possible rational roots are $\pm 1, \pm 2$, and we easily check that 1 is a root. Thus we factorize:

$$x^4+x^3-x^2+x-2 = (x-1)\,(x^3+2\,x^2+x+2). \tag{28}$$

Now possible roots for $x^3+2\,x^2+x+2$ are still $\pm 1, \pm 2$. This time we see that -2 is a root and finally we obtain

$$x^4+x^3-x^2+x-2 = (x-1)\,(x+2)\,(x^2+1) \tag{29}$$

which cannot be further factorized.

Now we write

$$\frac{-x^3+x^2-x+2}{(x-1)\,(x+2)\,(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C\,x+D}{x^2+1}. \tag{30}$$

Multiply both sides by $x-1$ and set $x=1$ we obtain $A=\frac{1}{6}$; Multiply both sides by $x+2$ and set $x=-2$ we obtain $B=-\frac{16}{15}$; Set $x=0$ we have $-1=-A+\frac{B}{2}+D \Longrightarrow D=-\frac{9}{30}$. Finally multiply both sides by $(x-1)\,(x+2)\,(x^2+1)$ and equate the x^3 term we have $-1=A+B+C \Longrightarrow C=-\frac{1}{10}$.

Finally we have

$$\int \frac{x^4}{x^4+x^3-x^2+x-2}\,\mathrm{d}x = x + \int\left[\frac{1/6}{x-1} + \frac{-16/15}{x+2} + \frac{-\frac{1}{10}\,x}{x^2+1} + \frac{-9/30}{x^2+1}\right]\mathrm{d}x \tag{31}$$

$$\begin{aligned} &= x + \frac{1}{6}\ln|x-1| - \frac{16}{15}\ln|x+2| \\ &\quad -\frac{1}{20}\ln(x^2+1) - \frac{9}{30}\arctan x + C. \end{aligned}\tag{32}$$

f) We factorize

$$\begin{aligned} x^6-1 &= (x-1)\,(x+1)\,(x^4+x^2+1) &(33)\\ &= (x-1)\,(x+1)\,(x^2+x+1)\,(x^2-x+1). &(34)\end{aligned}$$

Now write

$$\frac{1}{(x-1)\,(x+1)\,(x^2+x+1)\,(x^2-x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C\,x+D}{x^2+x+1} + \frac{E\,x+F}{x^2-x+1}. \tag{35}$$

We easily determine $A=\frac{1}{6}, B=-\frac{1}{6}$. To determine C—D we write

$$\begin{aligned}
1 &= A\,(x+1)\,(x^2+x+1)\,(x^2-x+1) \\
&\quad +B\,(x-1)\,(x^2+x+1)\,(x^2-x+1) \\
&\quad +(C\,x+D)\,(x^2-1)\,(x^2-x+1) \\
&\quad +(E\,x+F)\,(x^2-1)\,(x^2+x+1) \qquad (36)\\
&= \frac{1}{3}\,(x^4+x^2+1) \\
&\quad +[(C+E)\,x+(D+F)]\,(x^4-1) \\
&\quad +[(E-C)\,x+(F-D)]\,(x^3-x) \qquad (37)\\
&= (C+E)\,x^5+\left(\frac{1}{3}+(D+F)+(E-C)\right)x^4 \\
&\quad +(F-D)\,x^3+\left(\frac{1}{3}-(E-C)\right)x^2 \\
&\quad +(-(C+E)-(F-D))\,x+\left(\frac{1}{3}+(D+F)\right). \qquad (38)
\end{aligned}$$

Now comparing the coefficients for $1, x^2, x^3, x^5$ we obtain

$$\frac{1}{3}-(D+F) = 1; \qquad (39)$$

$$\frac{1}{3}-(E-C) = 0; \qquad (40)$$

$$F-D = 0; \qquad (41)$$

$$C+E = 0. \qquad (42)$$

We easily solve $D=F=-\frac{1}{3}$; $E=\frac{1}{6}, C=-\frac{1}{6}$. So we have the partial fraction resolution:

$$\frac{1}{x^6-1}=\frac{1}{6}\left[\frac{1}{x-1}-\frac{1}{x+1}-\frac{x+2}{x^2+x+1}+\frac{x-2}{x^2-x+1}\right] \qquad (43)$$

We have

$$\begin{aligned}
\int\frac{x+2}{x^2+x+1}\,\mathrm{d}x &\overset{t=x+1/2}{=\!=\!=} \int\frac{t+3/2}{t^2+3/4}\,\mathrm{d}t \\
&\overset{u=2t/\sqrt{3}}{=\!=\!=} \frac{4}{3}\int\frac{\frac{3}{4}\,u+\frac{3\sqrt{3}}{4}}{u^2+1}\,\mathrm{d}u \\
&= \frac{1}{2}\ln(u^2+1)+\sqrt{3}\arctan u+C \\
&= \frac{1}{2}\ln(x^2+x+1)+\sqrt{3}\arctan\left(\frac{2\,x+1}{\sqrt{3}}\right)+C. \qquad (44)
\end{aligned}$$

Similarly we have

$$\int\frac{x-2}{x^2-x+1}\,\mathrm{d}x=\frac{1}{2}\ln(x^2-x+1)-\sqrt{3}\arctan\left(\frac{2\,x-1}{\sqrt{3}}\right)+C. \qquad (45)$$

Summarizing, we have

$$\begin{aligned}
\int\frac{\mathrm{d}x}{x^6-1} &= \frac{1}{6}\ln|-1|-\frac{1}{6}\ln|x+1| \\
&\quad -\frac{1}{6}\left[\frac{1}{2}\ln(x^2+x+1)+\sqrt{3}\arctan\left(\frac{2\,x+1}{\sqrt{3}}\right)\right] \\
&\quad +\frac{1}{6}\left[\frac{1}{2}\ln(x^2-x+1)-\sqrt{3}\arctan\left(\frac{2\,x-1}{\sqrt{3}}\right)\right]+C. \qquad (46)
\end{aligned}$$

QUESTION 2. (5 PTS) *Let* P, Q *be polynomials with* $\deg(P) < \deg(Q)$. *Further assume that* $Q(x) = (x - a_1) \cdots (x - a_n)$ *for some* $a_1, \ldots, a_n \in \mathbb{R}$ *with* $\forall i \neq j, a_i \neq a_j$. *Prove*

$$\int \frac{P(x)}{Q(x)} \mathrm{d}x = \sum_{k=1}^{n} A_k \ln|x - a_k| + C \tag{47}$$

where $A_k = \frac{P(a_k)}{Q'(a_k)}$.

Proof. First we notice that

$$Q'(x) = (x - a_2) \cdots (x - a_n) + (x - a_1)(x - a_3) \cdots (x - a_n) + \cdots + (x - a_1) \cdots (x - a_{n-1}). \tag{48}$$

Thus for every $k \in \{1, 2, \ldots, n\}$, we have

$$\frac{Q(x)}{x - a_k}|_{x = a_k} = Q'(a_k). \tag{49}$$

Now we write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \cdots + \frac{A_n}{x - a_n}. \tag{50}$$

Multiply both sides by $x - a_k$ and then set $x = a_k$, we have $A_k = \frac{P(a_k)}{Q'(a_k)}$ as desired. $\square$

Remark. Alternatively, we can obtain A_k through L'Hospital:

$$A_k = \lim_{x \to a_k} \frac{(x - a_k) P(x)}{Q(x)} = \lim_{x \to a_k} \frac{P(x) + (x - a_k) P'(x)}{Q'(x)} = \frac{P(a_k)}{Q'(a_k)}. \tag{51}$$

Note that from (48) we see that $Q'(a_k) \neq 0$, and as Q' is a polynomial, $\lim_{x \to a_k} Q'(x) = Q'(a_k)$.