MATH 118 WINTER 2015 LECTURE 11 (JAN. 22, 2015)

- We have seen that rational functions $\frac{P(x)}{Q(x)}$ can in theory[1] always be integrated. Now we show that another large class of functions also enjoys this property.
- Birational functions of $\cos x$ and $\sin x$.
  - A polynomial of two variables is a sum of finitely many terms of the form $a\, x^k\, y^l$ where $a \in \mathbb{R}, k, l \in \mathbb{N} \cup \{0\}$, and x, y are the two variables.
  - A birational function (or simply a rational function of x, y) is a function of the form $\frac{P(x, y)}{Q(x, y)}$ where P, Q are both polynomials of x, y.
  - We claim that

$$\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)} \mathrm{d}x \tag{1}$$

    can always be reduced, through a change of variable, to the integration of a rational function of a single variable, and therefore such integrals can in theory always be calculated.
  - Examples.
    - $\int \tan x \, \mathrm{d}x$. Here $P(x, y) = y$, $Q(x, y) = x$.
    - $\int \cos^n x \, \mathrm{d}x$. Here $P(x, y) = x^n$, $Q(x, y) = 1$.
    - $\int \frac{1}{\sin^n x} \mathrm{d}x$. Here $P(x, y) = 1$, $Q(x, y) = y^n$.
    - $\int \cos^n x \sin^m x \, \mathrm{d}x$. Here $P(x, y) = x^n\, y^m$, $Q(x, y) = 1$.
- Integration of $\frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)}$ through the universal change of variable.
  - The universal change of variable is $t = \tan\left(\frac{x}{2}\right)$. We notice that

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{1 - t^2}{1 + t^2}; \tag{2}$$

$$\sin x = 2 \sin\frac{x}{2} \cos\frac{x}{2} = \frac{2\, t}{1 + t^2}; \tag{3}$$

$$\mathrm{d}x = \mathrm{d}(2 \arctan u) = \frac{2}{1 + t^2}. \tag{4}$$

    Thus under this change of variable we have

$$\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)} \mathrm{d}x = \int R(t)\, \mathrm{d}t \tag{5}$$

    where

$$R(x) = \frac{P\left(\frac{1 - t^2}{1 + t^2}, \frac{2\, t}{1 + t^2}\right)}{Q\left(\frac{1 - t^2}{1 + t^2}, \frac{2\, t}{1 + t^2}\right)} \frac{2}{1 + t^2} \tag{6}$$

1. and in practice

is rational.

Problem 1. Prove that $R(x)$ is rational.

- Examples.

Example 1. Calculate $\displaystyle\int \frac{\mathrm{d}x}{\cos x + \sin x}$.

Solution. We apply the change of variable $t = \tan\frac{x}{2}$. Then we have

$$\begin{aligned}\int \frac{\mathrm{d}x}{\cos x + \sin x} &= \int \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \frac{2}{1+t^2} \mathrm{d}t \\ &= \int \frac{2}{1+2t-t^2} \mathrm{d}t. \qquad (7)\end{aligned}$$

We solve this integral using partial fractions. Solve $1+2t-t^2=0$ gives $t_{1,2}=1\pm\sqrt{2}$. Therefore $1+2t-t^2 = \left(1+\sqrt{2}-t\right)\left(1-\sqrt{2}-t\right)$ and we write

$$\frac{2}{1+2t-t^2} = \frac{A}{t-\left(1+\sqrt{2}\right)} + \frac{B}{t-\left(1-\sqrt{2}\right)} \qquad (8)$$

and determine

$$A = -\frac{\sqrt{2}}{2}, \qquad B = \frac{\sqrt{2}}{2}. \qquad (9)$$

Therefore

$$\begin{aligned}\int \frac{2\,\mathrm{d}t}{1+2t-t^2} &= \frac{\sqrt{2}}{2} \int \left[\frac{1}{t-\left(1-\sqrt{2}\right)} - \frac{1}{t-\left(1+\sqrt{2}\right)}\right] \mathrm{d}t \\ &= \frac{\sqrt{2}}{2} \ln\left|\frac{t-\left(1-\sqrt{2}\right)}{t-\left(1+\sqrt{2}\right)}\right| + C. \qquad (10)\end{aligned}$$

Substituting back $u = \tan\left(\frac{x}{2}\right)$ we have

$$\int \frac{\mathrm{d}x}{\cos x + \sin x} = \frac{\sqrt{2}}{2} \ln\left|\frac{\tan\left(\frac{x}{2}\right) - \left(1-\sqrt{2}\right)}{\tan\left(\frac{x}{2}\right) - \left(1+\sqrt{2}\right)}\right| + C. \qquad (11)$$

Exercise 1. Calculate $\displaystyle\int \frac{\mathrm{d}x}{\cos x - \sin x}$.

Exercise 2. Calculate $\displaystyle\int \frac{\mathrm{d}x}{\cos^2 x + \sin x}$.

Example 2. Calculate $\displaystyle\int \frac{\mathrm{d}x}{1+2\cos x}$.

Solution. We have $P(x,y)=1$, $Q(x,y)=1+2y$. Thus the substitution $t=\tan\frac{x}{2}$ gives

$$\int \frac{\mathrm{d}x}{1+2\cos x} = \int \frac{1}{1+2\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} \mathrm{d}t = \int \frac{2}{3-t^2} \mathrm{d}t. \qquad (12)$$

Apply the method of partial fractions, we have

$$\int \frac{2}{3-t^2} \mathrm{d}t = \frac{1}{\sqrt{3}} \left[\int \frac{\mathrm{d}t}{\sqrt{3}-t} + \int \frac{\mathrm{d}t}{\sqrt{3}+t}\right] = \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C. \qquad (13)$$

Substituting back $t = \tan\frac{x}{2}$, we have

$$\int \frac{\mathrm{d}x}{1+2\cos x} = \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right| + C. \tag{14}$$

Exercise 3. Calculate $\int \frac{\mathrm{d}x}{1+2\sin x+3\cos x}$.

- Special cases.
  - The universal change of variable always works, but may not be the most efficient approach.

Example 3. Calculate $\int \frac{\sin 2x}{\sin^2 x+\cos x}\mathrm{d}x$.

Solution. Let $t = \cos x$, then we have

$$\begin{aligned}
\int \frac{\sin 2x}{\sin^2 x+\cos x}\mathrm{d}x &= -2\int \frac{t\,\mathrm{d}t}{1+t-t^2} \\
&= 2\int \frac{t}{\left(t-\frac{1+\sqrt{5}}{2}\right)\left(t-\frac{1-\sqrt{5}}{2}\right)}\mathrm{d}t \\
&= \int \left[\frac{1+\frac{1}{\sqrt{5}}}{t-\frac{1+\sqrt{5}}{2}} + \frac{1-\frac{1}{\sqrt{5}}}{t-\frac{1-\sqrt{5}}{2}}\right]\mathrm{d}t \\
&= \left(1+\frac{1}{\sqrt{5}}\right)\ln\left|t-\frac{1+\sqrt{5}}{2}\right| + \left(1-\frac{1}{\sqrt{5}}\right)\ln\left|t-\frac{1-\sqrt{5}}{2}\right| + C \\
&= \left(1+\frac{1}{\sqrt{5}}\right)\ln\left|\cos x-\frac{1+\sqrt{5}}{2}\right| + \left(1-\frac{1}{\sqrt{5}}\right)\ln\left|\cos x - \frac{1-\sqrt{5}}{2}\right| + C \\
&= \ln|1+\cos x-\cos^2 x| + \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+1-2\cos x}{\sqrt{5}-1+2\cos t}\right| + C.
\end{aligned} \tag{15}$$

Remark 4. To compare, let's try the universal change of variable $t = \tan\left(\frac{x}{2}\right)$. We have

$$\begin{aligned}
\int \frac{\sin 2x}{\sin^2 x+\cos x}\mathrm{d}x &= \int \frac{2\sin x\cos x}{\sin^2 x+\cos x}\mathrm{d}x \\
&= \int \frac{2\frac{2t}{1+t^2}\frac{1-t^2}{1+t^2}}{\left(\frac{2t}{1+t^2}\right)^2+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t \\
&= \int \frac{4t(1-t^2)}{4t^2+1-t^4}\frac{2}{1+t^2}\mathrm{d}t.
\end{aligned} \tag{16}$$

We see that in this approach we have to deal with a much more complicated rational function.

  - The following are the most important special cases.

PROPOSITION 5. (SPECIAL CASES) *Let $R(x, y)$ be birational and such that*

a) $R(-x, y) = -R(x, y)$, *or*

b) $R(x, -y) = -R(x, y)$, *or*

c) $R(-x, -y) = R(x, y)$.

Then $\int R(\sin x, \cos x)\, \mathrm{d}x$ *can be integrated through* $t = \sin x$, $t = \cos x$, $t = \tan x$, *respectively.*

Proof. Let $R(x, y) = \frac{P(x, y)}{Q(x, y)}$ where P, Q are polynomials that share no common factor.

a) In this case we have

$$P(x, y) = R(x, y)\, Q(x, y) \tag{17}$$

and therefore

$$P(-x, y) = -R(x, y)\, Q(-x, y). \tag{18}$$

Putting the two together we have

$$P(x, y) - P(-x, y) = R(x, y)\, [Q(x, y) + Q(-x, y)]. \tag{19}$$

As P, Q are polynomials, they can be written as

$$P(x, y) = a_n(y)\, x^n + \cdots + a_0(y), \quad Q(x, y) = b_m(y)\, x^m + \cdots + b_0(y). \tag{20}$$

Now it is easy to check that

$$P(x, y) - P(-x, y) = x\, P_1(x^2, y), \quad Q(x, y) + Q(-x, y) = Q_1(x^2, y) \tag{21}$$

where P_1, Q_1 are polynomials.

Exercise 4. Finish the proof.

b) This part is left as exercise.

Exercise 5. Prove this part.

c) In this case we have $P(x, y) = R(x, y)\, Q(x, y)$ and $P(-x, -y) = R(x, y)\, Q(-x, -y)$. This gives

$$R(x, y) = \frac{P(x, y) + P(-x, -y)}{Q(x, y) + Q(-x, -y)}. \tag{22}$$

As $P(x, y), Q(x, y)$ consists of terms of the form $x^k y^l$, we see that all the terms with $k + l$ odd are cancelled in both numerator and denominator. Now notice that when $k + l$ is even, there always holds

$$x^k y^l = \left(\frac{y}{x}\right)^l x^{k+l} = \left(\frac{y}{x}\right)^l (x^2)^{(k+l)/2}. \tag{23}$$

Thus we have $P(x, y) + P(-x, -y) = P_1\left(\frac{y}{x}, x^2\right)$ and $Q(x, y) + Q(-x, -y) = Q_1\left(\frac{y}{x}, x^2\right)$ where P_1, Q_1 are polynomials.

Exercise 6. Finish the proof. □

- More examples of the special cases.

Example 6. Calculate $\int \frac{\cos^3 x}{1 + \sin^2 x}\, \mathrm{d}x$.

Solution. We can check that $P(x,\, y) = x^3$, $Q(x,\, y) = 1 + y^2$ which gives $R(-x,\, y) = -R(x, y)$ so that substitution $t = \sin x$ would work. But of course it is easy to observe that

$$\begin{aligned}\int \frac{\cos^3 x}{1+\sin^2 x}\,\mathrm{d}x &= \int \frac{\cos^2 x}{1+\sin^2 x}\,\mathrm{d}\sin x\\ &= \int \frac{1-\sin^2 x}{1+\sin^2 x}\,\mathrm{d}\sin x\\ &= \int \frac{1-t^2}{1+t^2}\,\mathrm{d}t \qquad (t=\sin x)\\ &= \int \left[-1+\frac{2}{1+t^2}\,\mathrm{d}t\right]\\ &= -t+2\arctan t+C\\ &= -\sin x+2\arctan(\sin x)+C. \end{aligned} \tag{24}$$

Example 7. Calculate $\int \cos^4 x\,\mathrm{d}x$.

Solution. We have $R(x, y) = x^4$. Clearly $R(x, y) = R(-x, -y)$. Thus we set $t = \tan x$ and obtain

$$\begin{aligned}\int \cos^4 x &= \int \cos^6 x\,\mathrm{d}\tan x\\ &= \int \frac{1}{(1+t^2)^3}\,\mathrm{d}t. \end{aligned} \tag{25}$$

To calculate this integral we apply integration by parts:

$$\begin{aligned}\arctan t &= \int \frac{\mathrm{d}t}{1+t^2}\\ &= \frac{t}{1+t^2}+2\int \frac{t^2}{(1+t^2)^2}\,\mathrm{d}t\\ &= \frac{t}{1+t^2}+2\arctan t-2\int \frac{\mathrm{d}t}{(1+t^2)^2}. \end{aligned} \tag{26}$$

Therefore

$$\int \frac{\mathrm{d}t}{(1+t^2)^2} = \frac{1}{2}\left[\frac{t}{(1+t^2)}+\arctan t\right]+C. \tag{27}$$

Now we integrate by parts again:

$$\begin{aligned}\int \frac{\mathrm{d}t}{(1+t^2)^2} &= \frac{t}{(1+t^2)^2}+4\int \frac{t^2}{(1+t^2)^3}\,\mathrm{d}t\\ &= \frac{t}{(1+t^2)^2}+4\int \frac{\mathrm{d}t}{(1+t^2)^2}-4\int \frac{\mathrm{d}t}{(1+t^2)^3}. \end{aligned} \tag{28}$$

Therefore

$$\begin{aligned}\int \frac{\mathrm{d}t}{(1+t^2)^3} &= \frac{1}{4}\left[3\int \frac{\mathrm{d}t}{(1+t^2)^2}+\frac{t}{(1+t^2)^2}\right]\\ &= \frac{3\,t}{8\,(1+t^2)}+\frac{t}{4\,(1+t^2)^2}+\frac{3}{8}\arctan t+C. \end{aligned} \tag{29}$$

Substituting back $t = \tan x$, we finally arrive at

$$\begin{aligned}\int \cos^4 x \, \mathrm{d}x &= \frac{3}{8} \tan x \cos^2 x + \frac{1}{4} \tan x \cos^4 x + \frac{3}{8} x + C \\ &= \frac{3}{8} \sin x \cos x + \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} x + C. \end{aligned} \tag{30}$$

Exercise 7. Calculate $\int \sin^4 x \, \mathrm{d}x$ using $t = \tan x$.