Math 118 Winter 2015 Lecture 9 (Jan. 19, 2015)

- More examples.

Example 1. Calculate $\int \frac{x^2+2}{(x+1)^2\,(x-2)}\,\mathrm{d}x$.

Solution. We write

$$\frac{x^2+2}{(x+1)^2\,(x-2)} = \frac{A}{(x-2)} + \frac{B}{x+1} + \frac{C}{(x+1)^2}. \tag{1}$$

Multiply both sides by $x-2$ and set $x=2$, we see that $A=\frac{2}{3}$. Now multiply both sides by $(x+1)^2$ and then set $x=-1$, we have $C=-1$. Finally, multiply both sides by $(x+1)^2\,(x-2)$ we have

$$x^2+2 = A\,(x+1)^2 + B\,(x+1)\,(x-2) + C\,(x-2). \tag{2}$$

Equating the coefficients for x^2: $1 = A + B \Longrightarrow B = \frac{1}{3}$.

Finally

$$\begin{aligned} \int \frac{x^2+2}{(x+1)^2\,(x-2)}\,\mathrm{d}x &= \int \left[\frac{2/3}{x-2} + \frac{1/3}{x+1} + \frac{-1}{(x+1)^2}\right]\mathrm{d}x \\ &= \frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + \frac{1}{x+1} + C. \end{aligned} \tag{3}$$

Example 2. Calculate $\int \frac{\mathrm{d}x}{x^2-6\,x+5}$.

Solution. What is new here is that $Q(x) = x^2 - 6\,x + 5$ is not factorized. However as Q is quadratic, this is not a problem. We easily solve

$$x^2 - 6\,x + 5 = 0 \Longrightarrow x = 1, 5 \Longrightarrow x^2 - 6\,x + 5 = (x-1)\,(x-5). \tag{4}$$

Thus we have

$$\int \frac{\mathrm{d}x}{x^2-6\,x+5} = \frac{1}{4}\int \left[\frac{1}{x-5} - \frac{1}{x-1}\right]\mathrm{d}x = \frac{1}{4}\ln\left|\frac{x-5}{x-1}\right| + C. \tag{5}$$

Exercise 1. Calculate $\int \frac{\mathrm{d}x}{x^2+3\,x+1}$.

Exercise 2. Calculate $\int \frac{x^3}{x^2+4\,x+8}$.

- Factorization of Q.

When Q is cubic or higher, factorizing by hand becomes difficult or even impossible. The following theorem helpls a bit.

Theorem 3. *Let $\frac{p}{q}$ with p, q co-prime, $q > 0$, solve $a_n\,x^n + \cdots + a_1\,x + a_0 = 0$ where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then $p |\, a_0$, $q |\, a_n$.*

Proof. We have

$$a_n \left(\frac{p}{q}\right)^n + \cdots + a_1\,\frac{p}{q} + a_0 = 0. \tag{6}$$

Multiply both sides by q^n we reach

$$a_n\,p^n + \cdots + a_1\,p\,q^{n-1} + a_0\,q^n = 0. \tag{7}$$

This gives $p|\, a_0\, q^n$. As $(p,\, q) = 1$, we have $(p,\, q^n) = 1$ and therefore there must hold $p|\, a_0$. Similarly $q|\, a_n$. □

Example 4. Prove $\sqrt[4]{7} \notin \mathbb{Q}$.

Proof. Let $\alpha := \sqrt[4]{7}$. Then α solves $x^4 - 7 = 0$. Now assume $\alpha = \frac{p}{q}$. By the above theorem we have $p| -7, q|\, 1$. Therefore one of the following must hold:

$$\alpha = \frac{7}{1}, \quad \alpha = \frac{-7}{1}. \tag{8}$$

But as $1^4 < 7, 0^4 < 7, (-1)^4 < 7, n^4 > 7$ for all $n \neq -1, 0, 1$, clearly $\alpha \notin \mathbb{Z}$. Contradiction. □

Exercise 3. Prove that $\sqrt{2} + \sqrt[4]{3} \notin \mathbb{Q}$.

- More examples.

Example 5. Calculate $\int \frac{x^2 - 2\,x - 5}{x^3 + 6\,x^2 + 11\,x + 6}\, \mathrm{d}x$.

Solution. First check

$$\deg(x^2 - 2\,x - 5) < \deg(x^3 + 6\,x^2 + 11\,x + 6). \tag{9}$$

Next we factorize. Let $\frac{p}{q}$ solve $x^3 + 6\ x^2 + 11\ x + 6 = 0$. Then by Theorem 3 the only possibilities are $q = 1, p = \pm 1, \pm 2, \pm 3, \pm 6$. Clearly 1 is not a solution by -1 is. Thus we have $(x+1)|\,(x^3 + 6\,x^2 + 11\,x + 6)$ and perform division to obtain

$$x^3 + 6\,x^2 + 11\,x + 6 = (x+1)\,(x^2 + 5\,x + 6) = (x+1)\,(x+2)\,(x+3). \tag{10}$$

Now write

$$\frac{x^2 - 2\,x - 5}{(x+1)\,(x+2)\,(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}. \tag{11}$$

The usual procedure gives

$$A = -1, \quad B = -3, \quad C = 5. \tag{12}$$

Therefore

$$\int \frac{x^2 - 2\,x - 5}{x^3 + 6\,x^2 + 11\,x + 6}\, \mathrm{d}x = -\ln|x+1| - 3\ln|x+2| + 5\ln|x+3| + C. \tag{13}$$

Example 6. Calculate $\int \frac{\mathrm{d}x}{x^3 - 1}$.

Solution. Clearly 1 solves $x^3 - 1 = 0$. So we have

$$x^3 - 1 = (x-1)\,(x^2 + x + 1). \tag{14}$$

As $x^2 + x + 1 = 0$ has no real solution, the factorization is complete. Now we write

$$\frac{1}{(x-1)\,(x^2 + x + 1)} = \frac{A}{x-1} + \frac{B\,x + C}{x^2 + x + 1}. \tag{15}$$

Multiply by $x - 1$ and set $x = 1$ we have $A = \frac{1}{3}$. Setting $x = 0$ we have $-1 = -A + C \Longrightarrow C = -\frac{2}{3}$. Finally multiply both sides by $(x-1)\,(x^2 + x + 1)$ and compare the x^2 term, we have

$$0 = A + B \Longrightarrow B = -\frac{1}{3}. \tag{16}$$

Thus we have

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^3-1} &= \int \frac{1/3}{x-1}\,\mathrm{d}x - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{(x+1/2)^2+\left(\sqrt{3}/2\right)^2}\,\mathrm{d}x \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{3}\,\frac{4}{3}\int \frac{x+2}{\left(\frac{x+1/2}{\sqrt{3}/2}\right)^2+1}\,\mathrm{d}x \\
&\overset{t=\frac{x+1/2}{\sqrt{3}/2}}{=\!=\!=} \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{t+\sqrt{3}}{t^2+1}\,\mathrm{d}t \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|t^2+1| - \frac{\sqrt{3}}{3}\arctan t + C \\
&= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2\,x+1}{\sqrt{3}}\right) + C. \qquad (17)
\end{aligned}$$

Exercise 4. Calculate the following integrals.

$$\int \frac{x+3}{x^3-3\,x^2+4}\,\mathrm{d}x; \quad \int \frac{\mathrm{d}x}{x^3+1}; \quad \int \frac{\mathrm{d}x}{x^4-1}. \tag{18}$$

Exercise 5. Calculate the following integrals.

$$\int \frac{\mathrm{d}x}{x^4+1}; \quad \int \frac{\mathrm{d}x}{x^4+5\,x^2+4}; \quad \int \frac{\mathrm{d}x}{x^4+x^2+1}. \tag{19}$$