Math 118 Winter 2015 Lecture 8 (Jan. 16, 2015)

- Integration of rational functions.
  - Goal:

$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x. \tag{1}$$

  - Know: $a, b, A, B, C \in \mathbb{R},\ k \in \mathbb{N}$. How to do

$$\int \frac{A\,\mathrm{d}x}{(x-a)^k}, \qquad \int \frac{B\,x+C}{[(x-a)^2+b^2]^k}\,\mathrm{d}x. \tag{2}$$

Exercise 1. How?

  - Idea: There always holds

$$\frac{P(x)}{Q(x)} = P_0(x) + \sum \text{terms of the form } \frac{A}{(x-a)^k} + \sum \text{terms of the form } \frac{B\,x+C}{[(x-a)^2+b^2]^l}. \tag{3}$$

Where $P_0(x)$ is a polynomial and the k in the sums satisfy $(x-a)^k$, $[(x-a)^2+b^2]^l$ divide $Q(x)$.

Remark 1. We understand why (3) should hold through an analogy in number theory. Consider $q_1, q_2 \in \mathbb{N}$ co-prime. Then there are $c_1, c_2 \in \mathbb{Z}$ such that

$$c_1\,q_1 + c_2\,q_2 = 1. \tag{4}$$

Thus for every $p \in \mathbb{N}$, we have

$$\frac{p}{q} = p_0 + \frac{p_1}{q_1} + \frac{p_2}{q_2} \tag{5}$$

with $|p_1| < q_1, |p_2| < q_2$.

Exercise 2. Prove (4). (Hint:[1])

Exercise 3. Prove (5) using (4).

Exercise 4. Let $p, q \in \mathbb{N}$. Let $q = q_1^{k_1} \cdots q_l^{k_l}$ be the prime factorization of q. Then there are p_0, $p_{i,j} \in \mathbb{Z}$ such that

$$\frac{p}{q} = p_0 + \frac{p_{1,1}}{q_1} + \frac{p_{1,2}}{q_1^2} + \cdots + \frac{p_{1,k_1}}{q_1^{k_1}} + \cdots + \frac{p_{l,1}}{q_l} + \cdots + \frac{p_{l,k_l}}{q_l^{k_l}}. \tag{6}$$

and furthermore $|p_{1,1}|, \ldots, |p_{1,k_1}| < q_1, \ldots, |p_{l,1}|, \ldots, |p_{l,k_l}| < q_l$.

- Examples.

Example 2. Calculate $\displaystyle\int \frac{3\,x+2}{(x-1)\,(x-2)\,(x-3)}\,\mathrm{d}x$.

Solution. We write

$$\frac{3\,x+2}{(x-1)\,(x-2)\,(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \tag{7}$$

1. Recall Euclid's algorithm of finding the g.c.d. of two numbers.

To determine A, we multiply both sides by $(x-1)$ and then set $x=1$: $A=\frac{5}{2}$. Similarly we obtain $B=-8$, $C=\frac{11}{2}$. Thus

$$\int \frac{3\,x+2}{(x-1)\,(x-2)\,(x-3)}\,\mathrm{d}x = \int \frac{5/2}{x-1}\,\mathrm{d}x + \int \frac{-8}{x-2}\,\mathrm{d}x + \int \frac{11/2}{x-3}\,\mathrm{d}x = \frac{5}{2}\ln|x-1| - 8\ln|x-2| + \frac{11}{2}\ln|x-3| + C. \tag{8}$$

Example 3. Calculate $\int \frac{4\,x^3}{(x-1)\,(x-2)}\,\mathrm{d}x$.

Solution. As the degree of the numerator is no less than that of the denominator, we need to first calculate

$$4\,x^3 \div [(x-1)\,(x-2)] = 4\,(x+3) \ldots . 4\,(7\,x-6). \tag{9}$$

Thus

$$\frac{4\,x^3}{(x-1)\,(x-2)} = 4\,(x+3) + \frac{4\,(7\,x-6)}{(x-1)\,(x-2)}. \tag{10}$$

Now we write

$$\frac{4\,(7\,x-6)}{(x-1)\,(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}. \tag{11}$$

Similar to the previous example we have $A=-4, B=32$. Therefore

$$\int \frac{4\,x^3}{(x-1)\,(x-2)}\,\mathrm{d}x = \int 4\,(x+3)\,\mathrm{d}x - \int \frac{4}{x-1}\,\mathrm{d}x + \int \frac{32}{x-2}\,\mathrm{d}x = 2\,x^2 + 12\,x - 4\ln|x-1| + 32\ln|x-2| + C. \tag{12}$$

Example 4. Calculate $\int \frac{x+1}{(x-1)\,(x^2+1)^2}\,\mathrm{d}x$.

Solution. We write

$$\frac{x+1}{(x-1)\,(x^2+1)^2} = \frac{A}{x-1} + \frac{B\,x+C}{x^2+1} + \frac{D\,x+E}{(x^2+1)^2}. \tag{13}$$

Multiply both sides by $x-1$ and setting $x=1$ we have $A=1/2$. There is no simple way to determine B, C, D, E so we just do it the brute force way: Multiply both sides by $(x-1)\,(x^2+1)^2$, we reach

$$x+1 = A\,(x^2+1)^2 + (B\,x+C)\,(x-1)\,(x^2+1) + (D\,x+E)\,(x-1). \tag{14}$$

Comparing coefficients on both sides:

$$x^4 \ : \ 0 = A + B; \tag{15}$$

$$x^3 \ : \ 0 = -B + C; \tag{16}$$

$$x^2 \ : \ 0 = 2\,A + B - C + D; \tag{17}$$

$$1 \ : \ 1 = A - C - E. \tag{18}$$

With $A=1/2$, we solve

$$B=-1/2, \qquad C=-1/2, \qquad D=-1, \qquad E=0. \tag{19}$$

Therefore

$$\int \frac{x+1}{(x-1)\,(x^2+1)^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x-1}\,\mathrm{d}x - \frac{1}{2}\int \frac{x+1}{x^2+1}\,\mathrm{d}x - \int \frac{x}{(x^2+1)^2}\,\mathrm{d}x = \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x + \frac{1}{2}\,\frac{1}{(x^2+1)} + C. \tag{20}$$