MATH 118 WINTER 2015 HOMEWORK 2 SOLUTIONS

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QUESTION 1. (10 PTS) *Calculate the following indefinite integrals through change of variables. Please provide enough details.*

a) (3 PTS) $\int x^2\sqrt{1-x^2}\,\mathrm{d}x$;

b) (3 PTS) $\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$;

c) (4 PTS) $\int \frac{\mathrm{d}x}{\sqrt{1-\sin^4 x}}$.

Solution.

a) We set $x=\sin t$ and obtain

$$\begin{aligned}
\int x^2\sqrt{1-x^2}\,\mathrm{d}x &= \int \sin^2 t\cos^2 t\,\mathrm{d}t && (1)\\
&= \frac{1}{4}\int \sin^2 2\,t\,\mathrm{d}t && (2)\\
&= \frac{1}{8}\int [1-\cos 4\,t]\mathrm{d}t && (3)\\
&= \frac{t}{8}-\frac{1}{32}\sin 4\,t+C && (4)\\
&= \frac{1}{8}[t-\sin t\cos t\cos 2\,t]+C && (5)\\
&= \frac{1}{8}[t-\sin t\cos t\,(1-2\sin^2 t)]+C && (6)\\
&= \frac{1}{8}\left[\arcsin x-x\,(1-2\,x^2)\sqrt{1-x^2}\right]+C. && (7)
\end{aligned}$$

b) We set $x=\sin^2 t$ ($t\in[0,\pi/2)$ as x must satisfy $x\in[0,1)$).

$$\begin{aligned}
\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x &= \int \frac{\sin t}{\cos t}\,2\sin t\cos t\,\mathrm{d}t && (8)\\
&= 2\int \sin^2 t\,\mathrm{d}t && (9)\\
&= \int [1-\cos 2\,t]\,\mathrm{d}t && (10)\\
&= t-\frac{1}{2}\sin 2\,t+C && (11)\\
&= \arcsin x-\sqrt{x\,(1-x)}+C. && (12)
\end{aligned}$$

c) We have

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1-\sin^4 x}} &= \int \frac{\mathrm{d}x}{\cos^2 x \sqrt{\frac{1}{\cos^4 x} - \tan^4 x}} && (13)\\
&\overset{u=\tan x}{=\!=\!=} \int \frac{\mathrm{d}u}{\sqrt{(1+u^2)^2 - u^4}} && (14)\\
&= \int \frac{\mathrm{d}u}{\sqrt{1+2\,u^2}} && (15)\\
&\overset{v=\sqrt{2}\,u}{=\!=\!=} \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}v}{\sqrt{1+v^2}} && (16)\\
&\overset{v=\tan t}{=\!=\!=} \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}t}{\cos t} && (17)\\
&= \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}(\sin t)}{1-\sin^2 t} && (18)\\
&= \frac{1}{2\sqrt{2}} \ln \left| \frac{1+\sin t}{1-\sin t} \right| + C && (19)\\
&= \frac{1}{\sqrt{2}} \ln \frac{1+\sin t}{\cos t} + C && (20)\\
&= \frac{1}{\sqrt{2}} \ln \left(v + \sqrt{v^2+1}\right) + C && (21)\\
&= \frac{1}{\sqrt{2}} \ln \left(\sqrt{2}\tan x + \sqrt{2\tan^2 x + 1}\right) + C. && (22)
\end{aligned}
$$

Remark. Wolframalpha calculated the integral as

$$
\frac{\cos x \sqrt{\cos(2\,x) - 3}\, \arctan\left[\frac{2\sin x}{\sqrt{\cos(2\,x) - 3}}\right]}{2\sqrt{1-\sin^4 x}} + C. \tag{23}
$$

Not quite sure how to show the equivalence yet.

QUESTION 2. (10 PTS) *Calculate the following indefinite integrals through integration by parts (note change of variables may be needed at certain steps). Please provide enough details.*

a) (2 PTS) $\int \ln(1+x^2)\,\mathrm{d}x$;

b) (2 PTS) $\int x^2 e^{-x}\,\mathrm{d}x$;

c) (3 PTS) $\int \sqrt{x}\ln^2 x\,\mathrm{d}x$;

d) (3 PTS) $\int \frac{x}{\cos^2 x}\,\mathrm{d}x$;

Solution.

a) We have

$$
\begin{aligned}
\int \ln(1+x^2)\,\mathrm{d}x &= x\ln(1+x^2) - \int x\,\mathrm{d}\ln(1+x^2) && (24)\\
&= x\ln(1+x^2) - \int \frac{2\,x^2}{1+x^2}\,\mathrm{d}x && (25)\\
&= x\ln(1+x^2) - 2\,x + 2\arctan x + C. && (26)
\end{aligned}
$$

b) We have

$$\begin{aligned}\int x^2 e^{-x}\,\mathrm{d}x &= -\int x^2\,\mathrm{d}(e^{-x}) && (27)\\ &= -x^2 e^{-x} + 2\int x\, e^{-x}\,\mathrm{d}x && (28)\\ &= -x^2 e^{-x} - 2\int x\,\mathrm{d}e^{-x} && (29)\\ &= -x^2 e^{-x} - 2\,x\,e^{-x} + 2\int e^{-x}\,\mathrm{d}x && (30)\\ &= -(x^2+2\,x+2)\,e^{-x} + C. && (31)\end{aligned}$$

c) Set $x = t^2$. we have

$$\begin{aligned}\int \sqrt{x}\,\ln^2 x\,\mathrm{d}x &= 8\int t^2 \ln^2 t\,\mathrm{d}t && (32)\\ &= \frac{8}{3}\int \ln^2 t\,\mathrm{d}t^3 && (33)\\ &= \frac{8}{3}\left[t^3\ln^2 t - \int t^3\,\mathrm{d}\ln^2 t\right] && (34)\\ &= \frac{8}{3}t^3\ln^2 t - \frac{16}{3}\int t^2 \ln t\,\mathrm{d}t && (35)\\ &= \frac{8}{3}t^3\ln^2 t - \frac{16}{9}\int \ln t\,\mathrm{d}t^3 && (36)\\ &= \frac{8}{3}t^3\ln^2 t - \frac{16}{9}t^3\ln t + \frac{16}{9}\int t^2\,\mathrm{d}t && (37)\\ &= \frac{8}{3}t^3\ln^2 t - \frac{16}{9}t^3\ln t + \frac{16}{27}t^3 + C && (38)\\ &= \frac{8}{27}t^3[9\ln^2 t - 6\ln t + 2] + C && (39)\\ &= \frac{8}{27}x^{3/2}\left[\frac{9}{4}\ln^2 x - 3\ln x + 2\right] + C. && (40)\end{aligned}$$

d) We have

$$\begin{aligned}\int \frac{x}{\cos^2 x}\,\mathrm{d}x &= \int x\,\mathrm{d}(\tan x) && (41)\\ &= x\tan x - \int \tan x\,\mathrm{d}x && (42)\\ &= x\tan x + \int \frac{\mathrm{d}(\cos x)}{\cos x} && (43)\\ &= x\tan x + \ln|\cos x| + C. && (44)\end{aligned}$$