MATH 118 WINTER 2015 LECTURE 6 (JAN. 14, 2015)

- Integration by parts.
  - Observation: u, v differentiable, then
$$(u\,v)' = u'\,v + u\,v' \Longrightarrow u\,v' = (u\,v)' - u'\,v. \tag{1}$$
  - Taking indefinite integral:
$$\int u\,v'\,\mathrm{d}x = u\,v - \int v\,u'\,\mathrm{d}x. \tag{2}$$
  - Use differential symbol:
$$\int u\,\mathrm{d}v = u\,v - \int v\,\mathrm{d}u. \tag{3}$$

Exercise 1. Shall we use $u\,v + C$ instead of $u\,v$? Explain.

Exercise 2. Assume further that u' is continuous. Prove that $\int u\,v'\,\mathrm{d}x$ and $\int v\,u'\,\mathrm{d}x$ both exist. (Hint:[1])

Problem 1. Are there u, v differentiable such that $\int u\,v'\,\mathrm{d}x$ does not exist? (Hint:[2])

- Examples

Example 1. Calculate
$$\int x\,e^x\,\mathrm{d}x, \qquad \int \ln x\,\mathrm{d}x, \qquad \int \arcsin x\,\mathrm{d}x. \tag{4}$$

Solution. We have
$$\begin{aligned}\int x\,e^x\,\mathrm{d}x &= \int x\,\mathrm{d}e^x \\ &= x\,e^x - \int e^x\,\mathrm{d}x \\ &= (x-1)\,e^x + C. \end{aligned} \tag{5}$$
$$\begin{aligned}\int \ln x\,\mathrm{d}x &= x\ln x - \int x\,\mathrm{d}\ln x \\ &= x\ln x - \int \mathrm{d}x \\ &= x\,(\ln x - 1) + C. \end{aligned} \tag{6}$$
$$\begin{aligned}\int \arcsin x\,\mathrm{d}x &= x\arcsin x - \int x\,\mathrm{d}\arcsin x \\ &= x\arcsin x - \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\ &= x\arcsin x + \frac{1}{2}\int \frac{\mathrm{d}(1-x^2)}{\sqrt{1-x^2}} \\ &= x\arcsin x + \sqrt{1-x^2} + C. \end{aligned} \tag{7}$$

Example 2. Calculate
$$\int x\,e^{2x}\,\mathrm{d}x, \quad \int x^2\cos 3\,x\,\mathrm{d}x, \quad \int x^2\ln^2 x\,\mathrm{d}x, \quad \int \arctan x\,\mathrm{d}x, \quad \int x\arctan x\,\mathrm{d}x. \tag{8}$$

1. Let $F(x) \in \int v\,u'\,\mathrm{d}x$ (prove the existence of this first), then there must hold $[u\,v - F]' = u\,v'$.

2. Seems to me $u = x^2\sin(1/x^3)$, $v = x^2\cos(1/x^3)$ could work by trying to prove $\lim_{x\to 0+} F(x) = +\infty$ if $F' = u\,v'$. Please inform me if I am wrong.

Solution.

We have

$$\begin{aligned}\int x\, e^{2x}\, \mathrm{d}x &= \int x\, \mathrm{d}\left(\frac{e^{2x}}{2}\right)\\ &= \frac{1}{2}\, x\, e^{2x} - \int \frac{e^{2x}}{2}\, \mathrm{d}x\\ &= \left(\frac{x}{2} - \frac{1}{4}\right) e^{2x} + C. \qquad (9)\end{aligned}$$

$$\begin{aligned}\int x^2 \cos 3\, x\, \mathrm{d}x &= \int x^2\, \mathrm{d}\frac{\sin 3\, x}{3}\\ &= x^2\, \frac{\sin 3\, x}{3} - \int \frac{\sin 3\, x}{3}\, 2\, x\, \mathrm{d}x\\ &= \frac{x^2 \sin 3\, x}{3} + \int \frac{2\, x}{3}\, \mathrm{d}\left(\frac{\cos 3\, x}{3}\right)\\ &= \frac{x^2 \sin 3\, x}{3} + \frac{2\, x \cos 3\, x}{9} - \int \frac{2 \cos 3\, x}{9}\, \mathrm{d}x\\ &= \frac{x^2 \sin 3\, x}{3} + \frac{2\, x \cos 3\, x}{9} - \frac{2}{27} \sin 3\, x + C. \qquad (10)\end{aligned}$$

$$\begin{aligned}\int x^2 \ln^2 x\, \mathrm{d}x &= \int \ln^2 x\, \mathrm{d}\left(\frac{x^3}{3}\right)\\ &= \frac{x^3}{3} \ln^2 x - \int \frac{x^3}{3}\, 2 \ln x\, \frac{1}{x}\, \mathrm{d}x\\ &= \frac{x^3}{3} \ln^2 x - \int \frac{2\, x^2}{3} \ln x\, \mathrm{d}x\\ &= \frac{x^3}{3} \ln^2 x - \int \ln x\, \mathrm{d}\left(\frac{2}{9}\, x^3\right)\\ &= \frac{x^3}{3} \ln^2 x - \frac{2\, x^3}{9} \ln x + \int \frac{2}{9}\, x^2\, \mathrm{d}x\\ &= \frac{x^3}{3} \ln^2 x - \frac{2}{9}\, x^3 \ln x + \frac{2}{27}\, x^3 + C\\ &= \frac{x^3}{3}\left[\ln^2 x - \frac{2}{3} \ln x + \frac{2}{9}\right] + C. \qquad (11)\end{aligned}$$

$$\begin{aligned}\int \arctan x\, \mathrm{d}x &= x \arctan x - \int x\, \mathrm{d}\arctan x\\ &= x \arctan x - \int \frac{x\, \mathrm{d}x}{x^2+1}\\ &= x \arctan x - \frac{1}{2} \int \frac{\mathrm{d}(x^2+1)}{x^2+1}\\ &= x \arctan x - \frac{1}{2} \ln(x^2+1) + C. \qquad (12)\end{aligned}$$

$$\begin{aligned}\int x \arctan x\, \mathrm{d}x &= \int \arctan x\, \mathrm{d}\left(\frac{x^2}{2}\right)\\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{x^2+1}\, \mathrm{d}x\\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left[1 - \frac{1}{1+x^2}\right] \mathrm{d}x\end{aligned}$$

$$= \frac{x^2+1}{2}\arctan x - \frac{x}{2} + C. \tag{13}$$