Math 118 Winter 2015 Lecture 5 (Jan. 12, 2015)

$$\int \frac{\mathrm{d(cabin)}}{\mathrm{cabin}} = \log(\mathrm{cabin}) + C = \mathrm{houseboat}. \tag{1}$$

- Recall

  - Type I substitution.

  $$\int f(x)\,\mathrm{d}x = \int f_1(u(x))\,u'(x)\,\mathrm{d}x = \int f_1(u)\,\mathrm{d}u = F_1(u) + C = F_1(u(x)) + C; \tag{2}$$

  Here $f(x) = f_1(u(x))\,u'(x)$.

  - Type II substitution.

  $$\int f(x)\,\mathrm{d}x \overset{x=x(t)}{=\!=\!=} \int f(x(t))\,x'(t)\,\mathrm{d}t = G(t) + C = G(T(x)) + C. \tag{3}$$

  Here $T(x)$ is the inverse function to $x(t)$.

- Examples.

Example 1. Calculate $\int \cos^n x\,\mathrm{d}x$ for $n = 1, 2, 3, 4$.

Solution.

- $n = 1$. This is in the table. $\int \cos x\,\mathrm{d}x = \sin x + C$.
- $n = 2$. We have

$$\int \cos^2 x\,\mathrm{d}x = \int \frac{\cos 2x + 1}{2}\,\mathrm{d}x = \frac{x}{2} + \frac{\sin 2x}{4} + C. \tag{4}$$

- $n = 3$. We have

$$\int \cos^3 x\,\mathrm{d}x = \int (1 - \sin^2 x)\,\mathrm{d}\sin x = \sin x - \frac{\sin^3 x}{3} + C. \tag{5}$$

- $n = 4$. We have

$$\begin{aligned}\int \cos^4 x\,\mathrm{d}x &= \int \left(\frac{1 + \cos 2x}{2}\right)^2 \mathrm{d}x \\ &= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,\mathrm{d}x \\ &= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right)\mathrm{d}x \\ &= \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \end{aligned} \tag{6}$$

Exercise 1. Calculate $\int \cos^n x\,\mathrm{d}x$ for $n = 5, 6$.

Example 2. Calculate $\int \frac{1}{\cos^n x}\,\mathrm{d}x$ for $n = 1, 2, 3, 4$.

Solution.

- $n=1$. We have (see lecture note on Jan. 8 for details of the last step).

$$\int \frac{\mathrm{d}x}{\cos x} = \int \frac{\cos x\,\mathrm{d}x}{1-\sin^2 x} = \int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C. \tag{7}$$

- $n=2$. This is in the table.

$$\int \frac{\mathrm{d}x}{\cos^2 x} = \tan x + C. \tag{8}$$

- $n=3$. We have

$$\int \frac{\mathrm{d}x}{\cos^3 x} = \int \frac{\mathrm{d}(\sin x)}{(1-\sin^2 x)^2} \overset{u=\sin x}{=\!=} \int \frac{\mathrm{d}u}{(1-u)^2(1+u)^2}. \tag{9}$$

  To evaluate the last integral we need the method of partial fractions. We will leave this task for a later lecture.

- $n=4$. We have

$$\int \frac{\mathrm{d}x}{\cos^4 x} = \int \frac{1}{\cos^2 x}\,\mathrm{d}\tan x \overset{u=\tan x}{=\!=} \int (1+u^2)\,\mathrm{d}u = \tan x + \frac{\tan^3 x}{3} + C. \tag{10}$$

Example 3. Calculate $\int \sqrt{1-x^2}\,\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int \sqrt{1-x^2}\,\mathrm{d}x &\overset{x=\sin t}{=\!=} \int \sqrt{1-\sin^2 t}\,\mathrm{d}\sin t \\ &= \int \cos^2 t\,\mathrm{d}t \\ &= \frac{t+\sin t\cos t}{2} + C \\ &= \frac{\arcsin x + x\sqrt{1-x^2}}{2} + C.\end{aligned} \tag{11}$$

Example 4. Calculate $\int \sqrt{1+x^2}\,\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int \sqrt{1+x^2}\,\mathrm{d}x &\overset{x=\frac{e^t-e^{-t}}{2}}{=\!=} \int \left(\frac{e^t+e^{-t}}{2}\right)^2 \mathrm{d}t \\ &= \int \left(\frac{e^{2t}+2+e^{-2t}}{4}\right)\mathrm{d}t \\ &= \frac{1}{8}[e^{2t}-e^{-2t}+4t] + C.\end{aligned} \tag{12}$$

The task now is to figure out $x = \frac{e^t - e^{-t}}{2} \Longrightarrow t = ?(x)$. Let $u = e^t$. Then we have $2x = u - \frac{1}{u}$ which means $u^2 - 2xu - 1 = 0$. Solving the equation gives $u = x \pm \sqrt{x^2+1}$. Recalling $u = e^t > 0$, we see that $u = x + \sqrt{x^2+1}$ and consequently $t = \ln\big(x + \sqrt{x^2+1}\big)$. Substituting into (12) we have the final answer as

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{x^2+1} + \ln\big(x+\sqrt{x^2+1}\big)\right] + C. \tag{13}$$

Exercise 2. Try to calculate $\int \sqrt{1+x^2}\,\mathrm{d}x$ using the change of variable $x = \tan t$.

Exercise 3. Calculate $\int \sqrt{x^2-1}\,\mathrm{d}x$.