Math 118 Winter 2015 Homework 1 Solutions

Due Thursday Jan. 15 3pm in Assignment Box

Question 1. (5 pts) *Let $F(x)$ be differentiable on (a, b) and let $c \in (a, b)$. Assume that both $\lim_{x \to c+} F'(x), \lim_{x \to c-} F'(x)$ exist and are finite. Prove that the two limits are equal.*

Proof. As $F(x)$ is differentiable at c, we have

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c} = F'(c). \tag{1}$$

On the other hand, let $x \in (c, b)$ be arbitrary. As $[c, x] \subset (a, b)$, we see that $F(x)$ is continuous on $[c, x]$ and differentiable on (c, x), therefore we can apply MVT to obtain

$$\frac{F(x) - F(c)}{x - c} = F'(\xi) \tag{2}$$

where $\xi \in (c, x)$. Now denote $m := \lim_{x \to c+} F'(x)$ and let $\varepsilon > 0$ be arbitrary. There is $\delta > 0$ such that for all $x \in (c, c + \delta)$, $|F'(x) - m| < \varepsilon$. Consequently, for every $x \in (c, c + \delta)$, we have

$$\left| \frac{F(x) - F(c)}{x - c} - m \right| = |F'(\xi) - m| < \varepsilon. \tag{3}$$

Thus by definition, we have proved

$$\lim_{x \to c+} \frac{F(x) - F(c)}{x - c} = m. \tag{4}$$

But following (1) we see that $\lim_{x \to c+} \frac{F(x) - F(c)}{x - c} = F'(c)$. Consequently $\lim_{x \to c+} F'(x) = F'(c)$. Similarly we have prove $\lim_{x \to c-} F'(x) = F'(c)$ and the proof ends. $\square$

Question 2. (5 pts) *Let $a, b \in \mathbb{R}$, $b \neq 0$, and $\int f(x)\, \mathrm{d}x = F(x) + C$. Calculate $\int a\, f(b\, x)\, \mathrm{d}x$ and justify your result.*

Solution. Set $u(x) := b\, x$, we have

$$\int a\, f(b\, x)\, \mathrm{d}x = \int \frac{a}{b}\, f(u(x))\, u'(x)\, \mathrm{d}x = \frac{a}{b}\, F(u(x)) + C = \frac{a}{b}\, F(b\, x) + C. \tag{5}$$

To justify, we calculate

$$\left(\frac{a}{b}\, F(b\, x) \right)' = \frac{a}{b}\, F'(b\, x)\, b = a\, f(b\, x). \tag{6}$$

Question 3. (10 pts) *Calculate the following indefinite integrals. Please provide enough details, in particular those about the substitutions you made.*

a) (2 pts) $\displaystyle\int \frac{(x+1)^3}{x}\, \mathrm{d}x$;

b) (2 pts) $\displaystyle\int \frac{x}{x^2 + 1}\, \mathrm{d}x$;

c) (2 pts) $\displaystyle\int \cot x\, \mathrm{d}x$;

d) (2 PTS) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x} + {}^3\sqrt{x}}$;

e) (2 PTS) $\displaystyle\int \frac{\mathrm{d}x}{1+\cos^2 x}$.

Solution.

a) We have

$$\int \frac{(x+1)^3}{x}\,\mathrm{d}x = \int \left[x^2 + 3\,x + 3 + \frac{1}{x}\right]\mathrm{d}x = \frac{x^3}{3} + \frac{3}{2}\,x^2 + 3\,x + \ln|x| + C. \tag{7}$$

b) Set $u(x) = x^2$. We have

$$\int \frac{x\,\mathrm{d}x}{x^2+1} = \frac{1}{2}\int \frac{\mathrm{d}u}{1+u} = \frac{1}{2}\ln|1+u| + C = \frac{1}{2}\ln(1+x^2) + C. \tag{8}$$

c) We have

$$\int \cot x\,\mathrm{d}x = \int \frac{\mathrm{d}(\sin x)}{\sin x} = \ln|\sin x| + C. \tag{9}$$

d) Set $x = t^6$. We have

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x} + {}^3\sqrt{x}} &= \int \frac{6\,t^5\,\mathrm{d}t}{t^3+t^2} \\
&= 6\int \frac{t^3}{1+t}\,\mathrm{d}t \\
&= 6\int \left[t^2 - t + 1 - \frac{1}{1+t}\right]\mathrm{d}t \\
&= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|1+t|\right] + C \\
&= 6\left[\frac{x^{1/2}}{3} - \frac{x^{1/3}}{2} + x^{1/6} - \ln\left|1 + x^{1/6}\right|\right] + C \\
&= 2\,x^{1/2} - 3\,x^{1/3} + 6\,x^{1/6} - 6\ln\left|1 + x^{1/6}\right| + C.
\end{aligned} \tag{10}$$

e) We have

$$\begin{aligned}
\int \frac{\mathrm{d}x}{1+\cos^2 x} &= \int \frac{\frac{1}{\cos^2 x}\,\mathrm{d}x}{1 + \frac{1}{\cos^2 x}} \\
&= \int \frac{\mathrm{d}(\tan x)}{2 + \tan^2 x} \\
&= \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C.
\end{aligned} \tag{11}$$