Math 118 Winter 2015 Lecture 3 (Jan. 8, 2015)

- Recall

$$\begin{aligned}
\int x^{\alpha}\,\mathrm{d}x &= \frac{1}{1+\alpha}\,x^{1+\alpha}+C \qquad \alpha\in\mathbb{R},\quad \alpha\neq 1; && (1)\\
\int \frac{\mathrm{d}x}{x} &= \ln|x|+C; && (2)\\
\int e^{x}\,\mathrm{d}x &= e^{x}+C; && (3)\\
\int \cos x\,\mathrm{d}x &= \sin x+C; && (4)\\
\int \sin x\,\mathrm{d}x &= -\cos x+C; && (5)\\
\int \frac{\mathrm{d}x}{(\cos x)^2} &= \tan x+C; && (6)\\
\int \frac{\mathrm{d}x}{(\sin x)^2} &= -\cot x+C; && (7)\\
\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} &= \arcsin x+C; && (8)\\
\int \frac{\mathrm{d}x}{1+x^2} &= \arctan x+C. && (9)
\end{aligned}$$

- Integration by substitution (change of variables)
  - Observation: If $F'=f$, then

$$F(u(x))'=f(u(x))\,u'(x). \tag{10}$$

Therefore, if $\int f(x)\,\mathrm{d}x=F(x)+C$, then $\int f(u(x))\,u'(x)\,\mathrm{d}x=F(u(x))+C$. This leads to

  - Type I change of variables:

To calculate $\int f(x)\,\mathrm{d}x$, we find appropriate $f_1(x), u(x)$ such that

$$f(x)=f_1(u(x))\,u'(x) \tag{11}$$

and furthermore $\int f_1(x)\,\mathrm{d}x=F_1(x)+C$ is easy to calculate. Then

$$\int f(x)\,\mathrm{d}x=F_1(u(x))+C. \tag{12}$$

Exercise 1. Justify this method.

Example 1. Calculate $\int \cos^3 x\,\mathrm{d}x$.

Solution. Set $u(x)=\sin x$ and $f_1(x)=1-x^2$. Then we have

$$\cos^3 x=f_1(u(x))\,u'(x). \tag{13}$$

As $\int (1-x^2)\,\mathrm{d}x=x-\frac{x^3}{3}+C$, we have

$$\int \cos^3 x\,\mathrm{d}x=\sin x-\frac{\sin^3 x}{3}+C. \tag{14}$$

Exercise 2. Check that (14) indeed holds.

- On the other hand, we could go "backwards" and have Type II change of variables:

  To calculate $\int f(x)\,\mathrm{d}x$, we find appropriate $u(t)$ such that an anti-derivative $F_1(t)$ of the function $f(u(t))\,u'(t)$ is easy to calculate. Then we have

$$\int f(x)\,\mathrm{d}x = F_1(T(x)) + C \tag{15}$$

  where $T(x)$ is the inverse function of $u(t)$.

  Exercise 3. Justify the method for the case $u'(t) > 0$ or <0 for all t,

Example 2. Calculate $\int e^{2x}\,\mathrm{d}x$.

Solution 1. We apply type I change of variable with $f_1(x) = e^x$ (which gives $F_1(x) = e^x$) and $u(x) = 2\,x$. Then we have

$$e^{2x} = \frac{1}{2}\,f_1(u(x))\,u'(x) \tag{16}$$

and consequently

$$\int e^{2x}\,\mathrm{d}x = \frac{1}{2}\,[F_1(u(x)) + C] = \frac{1}{2}\,e^{2x} + C. \tag{17}$$

Solution 2. We try type II change of variable. Set $u(t) = \frac{t}{2}$. Then we have

$$f_1(t) = f(u(t))\,u'(t) = \frac{1}{2}\,e^t. \tag{18}$$

Thus we have $F_1(t) = \frac{1}{2}\,e^t$ and

$$\int e^{2x}\,\mathrm{d}x = \frac{1}{2}\,e^{T(x)} + C = \frac{1}{2}\,e^{2x} + C. \tag{19}$$

With the help of integration by substitution, we could significantly expand the table of indefinite integrals.

Example 3. Calculate the following:

a) $\displaystyle\int \cos 2\,x\,\mathrm{d}x$;

b) $\displaystyle\int \frac{1}{\cos^2 3\,x}\,\mathrm{d}x$;

c) $\displaystyle\int \frac{\mathrm{d}x}{x-3}$;

d) $\displaystyle\int \frac{\mathrm{d}x}{(x-7)^5}$;

e) $\displaystyle\int x\,e^{-x^2/2}\,\mathrm{d}x$.

Solution.

a) We apply type I change of variables with $u(x) = 2\,x$ and $f_1(x) = \cos x$ to obtain

$$\int \cos 2\,x\,\mathrm{d}x = \frac{1}{2}\sin 2\,x + C. \tag{20}$$

b) We apply type I change of variables with $u(x)=3\,x$ and $f_1(x)=\frac{1}{\cos^2 x}$ to obtain

$$\int \frac{1}{\cos^2 3\,x}\,\mathrm{d}x=\frac{1}{3}\tan(3\,x)+C. \tag{21}$$

c) We apply type I change of variables with $u(x)=x-3$ and $f_1(x)=\frac{1}{x}$ to obtain

$$\int \frac{\mathrm{d}x}{x-3}=\ln|x-3|+C. \tag{22}$$

d) We apply type I change of variables with $u(x)=x-7$ and $f_1(x)=x^{-5}$ to obtain

$$\int \frac{\mathrm{d}x}{(x-7)^5}=-\frac{1}{4\,(x-7)^4}+C. \tag{23}$$

e) We apply type I change of variables with $u(x)=-x^2/2$ and $f_1(x)=-e^x$ to obtain

$$\int x\,e^{-x^2/2}\,\mathrm{d}x=-e^{-x^2/2}+C. \tag{24}$$

Exercise 4. Calculate a) – d) using type II change of variables.

Exercise 5. Try to apply type II change of variables to e). Is there any difficulty?

- We notice that in the table, two very natural integrals

$$\int \frac{\mathrm{d}x}{1-x^2}, \qquad \int \frac{\mathrm{d}x}{\sqrt{1+x^2}} \tag{25}$$

are missing.

Example 4. Calculate $\displaystyle\int \frac{\mathrm{d}x}{1-x^2}$.

Solution. We notice that

$$\frac{2}{1-x^2}=\frac{1}{x+1}-\frac{1}{x-1} \tag{26}$$

and therefore

$$\int \frac{\mathrm{d}x}{1-x^2}=\frac{1}{2}\left[\int \frac{\mathrm{d}x}{x+1}-\frac{\mathrm{d}x}{x-1}\right]=\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C. \tag{27}$$