MATH 118 WINTER 2015 LECTURE 2 (JAN. 7, 2015)

- Recall
  - the definition of indefinite integral: The indefinite integral for a function $f\colon (a,b) \mapsto \mathbb{R}$ is the following set:
  $$\{F(x) \mid F' = f \text{ on } (a,b)\}. \tag{1}$$
  Denoted $\int f(x)\,\mathrm{d}x$.
  - the structure of this set: If there is $F' = f$, then
  $$\int f(x)\,\mathrm{d}x = \{F(x) + C \mid C \in \mathbb{R}\}. \tag{2}$$
  Therefore we simply write $\int f(x)\,\mathrm{d}x = F(x) + C$.
- In contrast, $\int_a^b f(x)\,\mathrm{d}x$ is called the "definite integral".
  - "Indefinite integral" and "definite integral" are different:

Example 1. Let $f(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x < 0 \end{cases}$. Then there is no $F(x)\colon \mathbb{R} \mapsto \mathbb{R}$ such that $F' = f$ on $\mathbb{R}$.

Proof. Assume otherwise. Then there is $F(x)\colon \mathbb{R} \mapsto \mathbb{R}$ such that $F'(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x < 0 \end{cases}$. Let $x < 0$ be arbitrary. By MVT we have
$$\frac{F(x) - F(0)}{x - 0} = F'(c) = 0 \tag{3}$$
as $c \in (x, 0)$. Thus by definition
$$\lim_{x \to 0-} \frac{F(x) - F(0)}{x - 0} = 0, \tag{4}$$
a contradiction to $F'(0) = 1$, which means $\lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = 1$ and in particular $\lim_{x \to 0-} \frac{F(x) - F(0)}{x - 0} = 1$. □

Remark 2. As $f(x)$ is integrable on any $[a,b] \subset \mathbb{R}$, we see that it has definite integrals but no indefinite integral.

Exercise 1. Let $F(x)$ be differentiable on (a,b) and let $c \in (a,b)$. Prove that the following is impossible:

$\lim_{x \to c+} F'(x)$ and $\lim_{x \to c-} F'(x)$ both exist but are not equal.

In other words, derivatives cannot have "jump discontinuities".

Exercise 2. Does the above imply that all derivatives are continuous?

Problem 1. Let $f(x) = F'(x)$ on $\mathbb{R}$. Then $f(x)$ has the following "intermediate value property":

Let $a < b$ be arbitrary and let s be between $f(a)$ and $f(b)$. Then there is $c \in (a,b)$ such that $f(c) = s$.

Example 3. Let $F(x)=\begin{cases} x^2\sin\frac{1}{x^2} & x\neq 0\\ 0 & x=0\end{cases}$. Then $F(x)$ is differentiable on $\mathbb{R}$. Set $f(x)=F'(x)$. Then we have $\int f(x)\,\mathrm{d}x=F(x)+C$ but $f(x)$ is not Riemann integrable on any interval containing zero.

Remark 4. The reason why $f(x)$ is not Riemann integrable on such intervals is that $f(x)$ is unbounded around 0. On the other hand, there indeed are bounded functions that are derivatives, have indefinite integrals, but are not Riemann integrable. See for example: Goffman, Casper A bounded derivative which is not Riemann integrable. Amer. Math. Monthly 84 (1977), no. 3, 205–206.

- Thus, to evaluate $\int_a^b f(x)\,\mathrm{d}x$ our basic strategy is the following:
  1. Check that $f(x)$ is integrable on $[a,b]$;
  2. Try to calculate $\int f(x)\,\mathrm{d}x$, that is find $F(x)$ such that $F'(x)=f(x)$ on $(c,d)\supset[a,b]$;
  3. Calculate $F(b)-F(a)$.

  The key step is no doubt Step 2, the calculation of the indefinite integral.
- A table of simple indefinite integrals that **should be memorized**.

$$\int x^\alpha\,\mathrm{d}x = \frac{1}{1+\alpha}x^{1+\alpha}+C \qquad \alpha\in\mathbb{R},\quad \alpha\neq 1; \tag{5}$$

$$\int\frac{\mathrm{d}x}{x} = \ln|x|+C; \tag{6}$$

$$\int e^x\,\mathrm{d}x = e^x+C; \tag{7}$$

$$\int \cos x\,\mathrm{d}x = \sin x+C; \tag{8}$$

$$\int \sin x\,\mathrm{d}x = -\cos x+C; \tag{9}$$

$$\int \frac{\mathrm{d}x}{(\cos x)^2} = \tan x+C; \tag{10}$$

$$\int \frac{\mathrm{d}x}{(\sin x)^2} = -\cot x+C; \tag{11}$$

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x+C; \tag{12}$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \arctan x+C. \tag{13}$$

With this table we can calculate some simple indefinite integrals, thanks to the following theorem.

THEOREM 5. *Assume* $\int f(x)\,\mathrm{d}x=F(x)+C$, $\int g(x)\,\mathrm{d}x=G(x)+C$. *Then*

$$\int (f\pm g)(x)\,\mathrm{d}x=F(x)\pm G(x)+C. \tag{14}$$

Exercise 3. Prove Theorem 5.

Example 6. Calculate $\int \frac{(x-1)^2}{\sqrt{x}}\,\mathrm{d}x$.

Solution. We have

$$\int \frac{(x-1)^2}{\sqrt{x}}\,\mathrm{d}x = \int x^{3/2} - 2\,x^{1/2} + x^{-1/2}\,\mathrm{d}x = \frac{2}{5}\,x^{5/2} - \frac{4}{3}\,x^{3/2} + 2\,x^{1/2} + C. \tag{15}$$

Exercise 4. Calculate $\int \frac{(x-1)^2}{x}\,\mathrm{d}x$. (Ans:[1])

Example 7. Calculate $\int \tan^2 x\,\mathrm{d}x$.

Solution. We have

$$\int \tan^2 x\,\mathrm{d}x = \int \left[\frac{1}{\cos^2 x} - 1\right]\mathrm{d}x = \tan x - x + C. \tag{16}$$

Exercise 5. Calculate $\int \frac{\mathrm{d}x}{\sin^2 x \cos^2 x}$. (Ans: [2])

1. $\frac{1}{2}\,x^2 - 2\,x + \ln|x| + C$.
2. $\tan x - \cot x + C$.