Math 118 Winter 2015 Lecture 1 (Jan. 5, 2015)

- Recall we have three methods to evaluate an integral $\int_a^b f(x)\,\mathrm{d}x$.

    i. By definition;

    ii. First check that f is integrable on $[a,b]$, then try to find a family of partitions $\{P_n\}$ such that $U(f,P_n)$ (or $L(f,P_n)$) are easy to calculate. Then calculate $\lim_{n\to\infty}U(f, P_n)$;

    iii. Check that the three conditions for FTC1 are satisfied:

        - There is $F(x)$ such that $F'=f$ on (a,b);
        - This $F(x)$ is continuous on $[a,b]$;
        - f is integrable on $[a,b]$.

    Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a). \tag{1}$$

Exercise 1. Calculate $\int_0^1 \sin x\,\mathrm{d}x$ using Methods ii and iii.

- Indefinite integral.

Definition 1. *The indefinite integral for a function $f\colon(a,b)\mapsto\mathbb{R}$ is the following set:*

$$\{F(x)|\, F'=f \text{ on } (a,b)\}. \tag{2}$$

Denoted $\int f(x)\,\mathrm{d}x$.

Theorem 2. *Let $f(x)\colon(a,b)\mapsto\mathbb{R}$ be such that there is $F_0(x)$ satisfying $F_0'=f$ on (a,b). Then*

$$\int f(x)\,\mathrm{d}x = \{F_0(x)+C|\, C\in\mathbb{R}\}. \tag{3}$$

Remark 3. Thanks to this theorem we often simply write

$$\int f(x)\,\mathrm{d}x = F_0(x)+C. \tag{4}$$

Proof. (of Theorem 2)

- $\{F_0(x)+C\}\subseteq\int f(x)\,\mathrm{d}x$.

    We check

$$(F_0(x)+C)' = F_0'(x)+C' = f(x)+0 = f(x) \tag{5}$$

on (a,b). Therefore the claim holds.

- $\int f(x)\,\mathrm{d}x \subseteq \{F_0(x)+C\}$.

    Let $F(x)\in\int f(x)\,\mathrm{d}x$ be arbitrary. Let $c\in(a,b)$ be arbitrary and set $C := F(c)-F_0(c)$. Now for an arbitrary $x\in(a,b)$ different from c, by MVT we have the existence of $\xi\in(c,x)$ such that

$$\frac{[F(x)-F_0(x)]-[F(c)-F_0(c)]}{x-c} = F'(\xi)-F_0'(\xi) = f(\xi)-f(\xi) = 0 \tag{6}$$

which leads to $F(x) - F_0(x) = C$. As x is arbitrary, we have $F(x) = F_0(x) + C$ for this particular C for all $x \in (a, b)$. Therefore the claim holds.

Thus ends the proof. $\square$

Exercise 2. Check the the assumptions for MVT are satisfied, that is (6) indeed holds.