Math 117 Fall 2014 Lecture 46 (Nov. 28, 2014)

- Power series

Definition 1. *The formal sum* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is called a "power series".*

Let $c \in \mathbb{R}$. If the series $\sum_{n=0}^{\infty} a_n (c - x_0)^n$ converges to $L \in \mathbb{R}$, then the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ becomes a rule of mapping $c \mapsto L$. Therefore, if we let

$$A := \left\{ c \in \mathbb{R} | \sum_{n=0}^{\infty} a_n (c - x_0)^n \right\}, \tag{1}$$

then we can treat $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ as a function with domain A.

Example 2. Let $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series. Then $x_0 \in A$.

Example 3. Consider $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We notice that this is the Taylor series of e^x at $x_0 = 0$. Let $c \in \mathbb{R}$ be arbitrary, denote

$$s_n := \sum_{k=0}^{n} \frac{c^k}{k!}. \tag{2}$$

Applying Taylor's expansion with Lagrange form of remainder, we have

$$|e^c - s_n| = \left| \frac{e^{\xi}}{(n+1)!} c^{n+1} \right| < e^{|c|} \frac{|c|^{n+1}}{(n+1)!} \tag{3}$$

where $\xi \in (0, c)$.

Exercise 1. Let $c \in \mathbb{R}$ be arbitrary. Prove

$$\lim_{n \to \infty} \frac{|c|^n}{n!} = 0. \tag{4}$$

Now let $\varepsilon > 0$ be arbitrary. Let $N \in \mathbb{N}$ be such that for all $n > N$, $\frac{e^{|c|} |c|^{n+1}}{(n+1)!} < \varepsilon$. Then for such n we have $|e^c - s_n| < \varepsilon$. Thus by definition $\sum_{n=0}^{\infty} \frac{c^n}{n!} = e^c$.

From above we see that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for every $x \in \mathbb{R}$. Thus $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ defines a function with domain $\mathbb{R}$ and this function turns out to be e^x.

Example 4. Consider $\sum_{n=0}^{\infty} (x-1)^n$. It is clear that $|c - 1| < 1$ is necessary for convergence of $\sum_{n=0}^{\infty} (c-1)^n$. On the other hand, for such c we have

$$\sum_{k=0}^{n} (c-1)^k = \frac{1 - (c-1)^{n+1}}{2 - c} \longrightarrow \frac{1}{2 - c} \tag{5}$$

as $n \longrightarrow \infty$. Consequently $\sum_{n=0}^{\infty} (x-1)^n$ defines a function with domain $(-1, 1)$ and inside this interval it equals $\frac{1}{2-x}$.

Remark 5. Note that the domain of $\frac{1}{2-x}$ is larger than $(-1, 1)$.

Exercise 2. Prove that $\sum_{n=0}^{\infty} (x-1)^n$ is the Taylor series of $\frac{1}{2-x}$ at $x_0 = 1$.

- Radius of Convergence.

THEOREM 6. *Let* $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ *be an arbitrary power series. Define* $\rho := \left[\limsup_{n\to\infty} |a_n|^{1/n}\right]^{-1}$*. Then*

a) $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ *is convergent for* $|x - x_0| < \rho$*;*

b) $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ *is divergent for* $|x - x_0| > \rho$*.*

Proof.

a) As $|x - x_0| < \rho$, there is $\varepsilon > 0$ such that $r := |x - x_0|\left(\frac{1}{\rho} + \varepsilon\right) < 1$. Now by definition

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{\rho} \Longrightarrow \exists N \in \mathbb{N}, \forall n > N, \sup_{k\geqslant n}\left\{|a_k|^{1/k}\right\} < \frac{1}{\rho} + \varepsilon. \tag{6}$$

Therefore $\forall n > N, |a_n|^{1/n} < \frac{1}{\rho} + \varepsilon$. Consequently for $n > N$,

$$|a_n\,(x - x_0)^n| < r^n. \tag{7}$$

As $r \in (0,1)$ convergence of $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ follows.

b) Denote $\varepsilon := \frac{1}{\rho} - \frac{1}{|x - x_0|} > 0$. By definition

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{\rho} \Longrightarrow \exists N \in \mathbb{N}, \forall n > N, \sup_{k\geqslant n}\left\{|a_k|^{1/k}\right\} \geqslant \frac{1}{\rho} - \varepsilon = \frac{1}{|x - x_0|}. \tag{8}$$

Consequently $|a_n\,(x - x_0)^n| \geqslant 1$ and

$$\limsup_{n\to\infty} |a_n\,(x - x_0)^n| \geqslant 1 \tag{9}$$

which means $\lim_{n\to\infty} a_n\,(x - x_0)^n = 0$ does not hold and divergence follows. □

Remark 7. The situation ta $|x - x_0| = \rho$ is complicated. See the following examples.

Example 8. Consider $\sum_{n=0}^{\infty} x^n$. Then $a_n = 1$ and $\rho = 1$. We see that the series diverges at $x = \pm 1$.

Example 9. Consider $\sum_{n=0}^{\infty} \frac{x^n}{n}$. Then $a_n = \frac{1}{n}$ and $\rho = 1$.

Exercise 3. Prove $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1$.

- At $x = 1$ the series diverges;
- At $x = -1$ the series converges.

Example 10. Consider $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$. Then $a_n = \frac{1}{n^2}$ and $\rho = 1$. At $x = \pm 1$ the series still converge.