MATH 117 FALL 2014 LECTURE 45 (Nov. 27, 2014)

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Example 1. Prove that e is irrational.

Proof. Assume the contrary: $e=\frac{p}{q}$. Take $n>\max\{q,e\}$. We apply Taylor expansion with Lagrange form of remainder to e^x with $x_0=0$ and $x=1$:

$$\frac{p}{q}=e=1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}+\frac{e^c}{(n+1)!} \tag{1}$$

where $c\in(0,1)$. Multiply both sides by $n!$ we have

$$\frac{p}{q}\cdot(n!)=\left[1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}\right]\cdot(n!)+\frac{e^c}{n+1}. \tag{2}$$

As $n>q$, $q|\,n!$. Therefore $\frac{p}{q}\cdot(n!)\in\mathbb{Z}$. On the other hand clearly $\left[1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}\right](n!)\in\mathbb{Z}$. Therefore $\frac{e^c}{n+1}\in\mathbb{Z}$. However, as $c\in(0,1)$, $e^c\in(1,e)$ and $\frac{e^c}{n+1}\in\left(\frac{1}{n+1},\frac{e}{n+1}\right)\subset(0,1)$ and cannot be an integer. Contradiction. □

Example 2. Prove that e^2 is irrational.

Proof. Assume the contrary: $e^2=\frac{p}{q}$ with $q>0$, p, $q\in\mathbb{Z}$. This means $q\,e=p\,e^{-1}$. Apply Taylor expansion with Lagrange form of remainder to e^x with $x_0=0$ and $x=1$ for the left hand side and -1 for the righ hand side up to $n>q\,e+|p|$ and even:

$$q\left[1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}+\frac{e^{c_1}}{(n+1)!}\right]=p\left[\frac{1}{2}-\frac{1}{6}+\cdots+\frac{(-1)^n}{n!}+\frac{(-1)^{n+1}\,e^{c_2}}{(n+1)!}\right] \tag{3}$$

where $c_1\in(0,1), c_2\in(-1,0)$. Multiply both sides by $n!$ and re-arrange, we have

$$\left\{-q\left[1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{n!}\right]+p\left[\frac{1}{2}-\frac{1}{6}+\cdots+\frac{(-1)^n}{n!}\right]\right\}(n!)=\frac{q\,e^{c_1}+(-1)^{n+2}\,e^{c_2}}{n+1}:=R. \tag{4}$$

Thus $R\in\mathbb{Z}$. By our choice of n, we see that $R\in(0,1)$. Contradiction. □

Example 3. Estimate how well is $\frac{\pi}{4}$ approximated by $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$.

Solution. We expand $\arctan x$ at $x_0=0$ and $x=1$, up to degree $2\,n+2$:

$$\frac{\pi}{4}=\arctan 1=1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2\,n+1}+\frac{y^{(2n+3)}(c)}{(2\,n+3)!} \tag{5}$$

where $c\in(0,1)$ and $y(x):=\arctan x$. Recall that

$$y^{(2n+3)}(c)=(2\,n+2)!\,(\cos y)^{2n+3}\sin\!\left((2\,n+2)\left(y+\frac{\pi}{2}\right)\right) \tag{6}$$

we see that

$$\left|\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2\,n+1}\right)\right|<\frac{1}{2\,n+3}. \tag{7}$$

Exercise 1. Prove that this estimate is sharp, in the sense that there is $c>0$ such that there are infinitely many n satisfying

$$\left|\frac{\pi}{4}-\left(1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2\,n+1}\right)\right|\geqslant\frac{c}{2\,n+3}. \tag{8}$$

Remark 4. We see that calculating π through $4\left(1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2n+1}\right)$ is not efficient. It is much more efficient to take advantage if trig identities such as

$$\frac{\pi}{4}=\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right). \tag{9}$$