Math 117 Fall 2014 Lecture 41 (Nov. 19, 2014)

Read: **314** Differentiation §4.1.

- Higher Order Derivatives.

  Definition 1. *Let f be differentiable on (a, b). If $f'(x)$ is differentiable at some $c \in (a, b)$, we say f is twice differentiable at c and called $(f')'(c)$ its second order derivative at c. We denote it by $f''(c)$.*

  Notation 2. *We denote $f^{(1)} = f'$, $f^{(2)} = f''$.*

  Definition 3. *Let f be $(n-1)$-th differentiable on (a, b). If $f^{(n-1)}(x)$ is differentiable at $c \in (a, b)$, we say f is n-th differentiable at c and denote its n-th derivative at c as $f^{(n)}(c) := \left(f^{(n-1)}\right)'(c)$.*

  Notation 4. *For $n \leqslant 3$ we usually write f', f'', f''' for $f^{(1)}$, $f^{(2)}$, $f^{(3)}$.*

  Proposition 5. *Let f be k-th differentiable at c. Then for every $m, n \in \mathbb{N}$ with $m + n = k$, we have*

  $$f^{(k)}(c) = \left(f^{(m)}\right)^{(n)}(c). \tag{1}$$

  Proof. We prove by induction.

  - Base. $k = 2$. In this case $m = n = 1$ and the conclusion follows from Definition 1.
  - Induction. Assume the conclusion holds for $k = l$. Let $k = l + 1$ and $m, n \in \mathbb{N}$ be such that $m + n = l + 1$.
    - Case 1. $n = 1$. In this case the conclusion follows from Definition 3.
    - Case 2. $n > 1$. We have $n - 1 \in \mathbb{N}$ and by induction hypothesis

      $$f^{(k)}(c) = \left(f^{(k-1)}\right)'(c) = \left(\left(f^{(m)}\right)^{(n-1)}\right)'(c) = \left(f^{(m)}\right)^{(n)}(c) \tag{2}$$

      thanks to Definition 3. □

  Exercise 1. Let f, g be n-th differentiable at c. Prove

  - $f \pm g$ is n-th differentiable at c with

    $$(f \pm g)^{(n)}(c) = f^{(n)}(c) \pm g^{(n)}(c). \tag{3}$$

  - If $a \in \mathbb{R}$ then $a\, f$ is n-th differentiable at c with

    $$(a\, f)^{(n)}(c) = a\, f^{(n)}(c). \tag{4}$$

  - $f g$ is n-th differentiable at c;
  - If $g(c) \neq 0$, $\frac{f}{g}$ is n-th differentiable at c.

- Calculation of higher order derivatives.
  - Calculating a particular higher order derivative is simple.

    Example 6. Calculate $(\arctan x)'''$.

    Solution. We have

$$\begin{aligned}(\arctan x)''' &= [(\arctan x)']'' \\ &= \left(\frac{1}{1+x^2}\right)'' \\ &= \left(-\frac{2x}{(1+x^2)^2}\right)' \\ &= -\frac{2}{(1+x^2)^2}+2\frac{4x^2}{(1+x^2)^3} \\ &= \frac{6x^2-2}{(1+x^2)^3}. \end{aligned} \tag{5}$$

  - Things become subtle when we need to calculate derivative of a generic order n (or when we need to calculate a very high order of derivative).

    Example 7. Calculate $(\cos x)^{(n)}$.
    Solution. We observe

$$(\cos x)^{(n)} = \begin{cases} -\sin x & n=4k+1 \\ -\cos x & n=4k+2 \\ \sin x & n=4k+3 \\ \cos x & n=4k+4 \end{cases} = \cos\left(x-\frac{n\pi}{2}\right). \tag{6}$$

    The proof (through induction) of this formula is left as exercise.

    Exercise 2. Calculate $(x^k)^{(n)}$ for $n, k \in \mathbb{N}$. Justify.

    Exercise 3. Calculate $(e^{4x})^{(n)}$ for all $n \in \mathbb{N}$. Justify.

    Exercise 4. Calculate $(\ln(1+x))^{(n)}$ for all $n \in \mathbb{N}$. Justify.

    Example 8. Calculate $(\arctan x)^{(n)}$ at $x=0$.

    Solution.

    - Method 1. Let $y(x) := \arctan x$. We claim

$$y^{(n)}(x) = (n-1)!\cos^n y \sin\left(n\left(y+\frac{\pi}{2}\right)\right). \tag{7}$$

      Then we have $y^{(n)}(0) = (n-1)!\sin\left(-\frac{n\pi}{2}\right)$. We prove (7) by induction.

      - Base. We have

$$y'(x) = \frac{1}{1+x^2} = \cos^2 y = (1-1)!\cos^1 y \sin\left(1\cdot\left(y+\frac{\pi}{2}\right)\right). \tag{8}$$

- Induction. Assuming (7), we calculate

$$
\begin{aligned}
y^{(n+1)}(x) &= (n-1)!\Big[\cos^n y \sin\Big(n\Big(y+\frac{\pi}{2}\Big)\Big)\Big]' \\
&= (n-1)!\Big[n\cos^{n-1}y\,(-\sin y)\,y'\sin\Big(n\Big(y+\frac{\pi}{2}\Big)\Big) + \cos^n y\,n\cos\Big(n\Big(y+\frac{\pi}{2}\Big)\Big)\,y'\Big] \\
&= n!\cos^{n+1}y\Big[-\sin y\sin\Big(n\Big(y+\frac{\pi}{2}\Big)\Big) + \cos y\cos\Big(n\Big(y+\frac{\pi}{2}\Big)\Big)\Big] \\
&= n!\cos^{n+1}y\Big[\cos\Big(y+\frac{\pi}{2}\Big)\sin\Big(n\Big(y+\frac{\pi}{2}\Big)\Big) + \sin\Big(y+\frac{\pi}{2}\Big)\cos\Big(n\Big(y+\frac{\pi}{2}\Big)\Big)\Big] \\
&= n!\cos^{n+1}y\sin\Big[(n+1)\Big(y+\frac{\pi}{2}\Big)\Big]. \qquad (9)
\end{aligned}
$$

– Method 2.

THEOREM 9. (LEIBNIZ FORMULA) *We have*

$$(u\,v)^{(n)}(c) = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(c)\, v^{(n-k)}(c) \qquad (10)$$

if all derivatives involved exist.

***Exercise* 5.** *Prove this formula through induction.*

Applying Leibniz formula to

$$(1+x^2)\,y' = 1 \qquad (11)$$

we have

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (1+x^2)^{(k)}\, y^{(n-k)} = 0 \qquad (12)$$

which gives

$$y^{(n)} = -\sum_{k=1}^{n-1} \binom{n-1}{k} (1+x^2)^{(k)}\, y^{(n-k)}. \qquad (13)$$

Exercise 6. Calculate $(1+x^2)^{(k)}$ for a generic $k \in \mathbb{N}$.

Exercise 7. Complete the calculation of $(\arctan x)^{(n)}$ at $x=0$ using Method 2.

Exercise 8. Calculate $(\arcsin x)^{(n)}(0)$ for $n \in \mathbb{N}$. Justify.

Exercise 9. Calculate $(\tan x)^{(n)}$ for $n = 1, 2, ..., 8$.

Exercise 10. Let $x(t) := t - \sin t$, $y(t) := 1 - \cos t$.

a) Prove that $x(t) \mapsto y(t)$ defines a function $y = y(x)$.

b) Calculate $y''(0)$.