# MATH 117 FALL 2014 LECTURE 36 (Nov. 7, 2014)

**Read: Bowman** §4.B – §4.D.

- Maximizer and minimizer.

THEOREM 1. *Let $c \in (a, b)$ be a maximizer (or minimizer) of $f$ over $(a, b)$, that is for all $x \in (a, b)$, $f(x) \leqslant f(c)$. If $f$ is differentiable at $c$, then $f'(c) = 0$.*

**Proof.** Let $x_n < c$ be such that $\lim_{n \to \infty} x_n = c$. Then as $f$ is differentiable at $c$, we have $f'(c) = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c}$. Since $c$ is a maximizer, $f(x_n) \leqslant c$ which leads to $\frac{f(x_n) - f(c)}{x_n - c} \geqslant 0$. By Comparison Theorem we have $f'(c) = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geqslant 0$. Now let $x_n' > c$ be such that $\lim_{n \to \infty} x_n = c$. Similar consideration leads to $f'(c) = \lim_{n \to \infty} \frac{f(x_n') - f(c)}{x_n' - c} \leqslant 0$. Therefore $f'(c) = 0$. □

**Remark 2.** $f'(c) = 0$ does not necessarily imply $c$ being maximizer of minimizer, as the example $f(x) = x^3$ shows.

COROLLARY 3. *Let $f$ be differentiable on $(a, b)$. Then if $c \in (a, b)$ is a maximizer (or minimizer) of $f$ there holds $f'(c) = 0$.*

- Finding maximizer/minimizer of $f(x)$ over closed interval $[a, b]$. Assume $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

  1. Solve $f'(c) = 0$, get solutions $c_1, \ldots, c_k \in (a, b)$.
  2. Compare $f(a), f(b), f(c_1), \ldots, f(c_k)$. Find largest and smallest.

**Example 4.** Find maximum/minimum of $f(x) = x^3 - 3x^2 + 1$ over $[-1, 1]$.
**Solution.** Solving $0 = f'(x) = 3x^2 - 6x$ gives $c_1 = 0, c_2 = 2$. As $c_2 \notin (-1, 1)$ we discard it. Now compare $f(0) = 1$, $f(-1) = -3$, $f(1) = -1$ we see that the maximum is 1 and minimum is $-3$.

- Mean Value Theorem.

THEOREM 5. (ROLLE'S THEOREM) *Let $f$ satisfy*

1. *it is continuous on $[a, b]$;*
2. *it is differentiable on $(a, b)$;*
3. *$f(a) = f(b)$.*

*Then there is $c \in (a, b)$ such that $f'(c) = 0$.*

**Proof.** As $f$ is continuous on $[a, b]$ it reaches maximum and minimum on $[a, b]$. There are two cases.

- There is a maximizer or minimizer in $(a, b)$. Denote it by $c$. Then by Theorem 1 we have $f'(c) = 0$.
- The only maximizer/minimizer are $a, b$. Then we have $\forall x \in [a, b]$, $\min \{f(a), f(b)\} \leqslant f(x) \leqslant \max \{f(a), f(b)\}$. But $f(a) = f(b)$ so $f$ is constant on $(a, b)$ and consequently for every $c \in (a, b)$ we have $f'(c) = 0$. □

THEOREM 6. (MEAN VALUE THEOREM) *Let $f$ be*

1. *continuous on $[a, b]$,*

2. *differentiable on* $(a, b)$,

*then there is* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b) - f(a)}{b - a}$.

**Proof.** Set

$$h(x) := f(x) - \left[ f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right]. \tag{1}$$

We easily check that $h(x)$ satisfies the conditions for Rolle's Theorem. Consequently there is $c \in (a, b)$ such that $h'(c) = 0$ which is exactly $f'(c) = \frac{f(b) - f(a)}{b - a}$. □

Corollary 7. *Let* $f(x)$ *be differentiable on* $(a, b)$. *Then*

- $f$ *is increasing if and only if* $f'(x) \geqslant 0$ *everywhere on* $(a, b)$;
- $f$ *is decreasing if and only if* $f'(x) \leqslant 0$ *everywhere on* $(a, b)$;
- $f$ *is constant if and only if* $f'(x) = 0$ *everywhere on* $(a, b)$.

**Proof.** We prove the first claim and leave the other two as exercises.

- "Only if". Let $c \in (a, b)$ be arbitrary. As $f$ is increasing, for every $x < c$ we have $f(x) \leqslant f(c)$ and consequently $\frac{f(x) - f(c)}{x - c} \geqslant 0$. Taking limit $x \to c$ and applying Comparison Theorem, we see that $f'(c) \geqslant 0$.
- "If". Let $x, x' \in (a, b)$ be arbitrary with $x < x'$.

  1. As $f$ is differentiable on $(a, b)$, it is continuous on $(a, b)$ and in particular is continuous on $[x, x']$;
  2. As $f$ is differentiable on $(a, b)$ it is differentiable on $(x, x')$.

  Now apply MVT we have

$$\frac{f(x') - f(x)}{x' - x} = f'(c) \geqslant 0 \Longrightarrow f(x') \geqslant f(x). \tag{2}$$

Thus ends the proof. □

**Exercise 1.** Prove or disprove the following:

- If $f$ is strictly increasing on $(a, b)$, then $f'(c) > 0$ for every $c \in (a, b)$;
- If $f'(c) > 0$ for every $c \in (a, b)$, then $f$ is strictly increasing on $(a, b)$.