Math 117 Fall 2014 Lecture 35 (Nov. 6, 2014)

Read: Bowman §4.I, 4.J.

- Derivative of inverse function.

Theorem 1. *Let $f: [a, b] \mapsto \mathbb{R}$ be invertible, and satisfies*

  - *f is differentiable at $c \in (a, b)$;*
  - *$f'(c) \neq 0$;*
  - *f is continuous on $[a, b]$.*

Then its inverse function g is differentiable at $f(c)$ and furthermore $g'(f(c)) = \frac{1}{f'(c)}$.

Proof. It suffices to prove the following: Let $y_n \longrightarrow f(c)$ be arbitrary satisfying $\forall n \in \mathbb{N}$, $y_n \neq f(c)$, then

$$\lim_{n\to\infty} \frac{g(y_n) - g(f(c))}{y_n - f(c)} = \frac{1}{f'(c)}. \tag{1}$$

As f is invertible, there are unique x_n's such that $f(x_n) = y_n$ and $x_n \neq c$. Furthermore as f is continuous on $[a, b]$, so is g and consequently

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} g(f(x_n)) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} g(f(c)) = g(c). \tag{2}$$

Thus a consequence of $\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = f'(c)$ is $\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = f'(c)$. Finally, as $f'(c) \neq 0$ we have

$$\frac{1}{f'(c)} = \lim_{n\to\infty} \frac{x_n - c}{f(x_n) - f(c)} = \lim_{n\to\infty} \frac{g(y_n) - g(f(c))}{y_n - f(c)}. \tag{3}$$

Therefore $g'(f(c)) = \frac{1}{f'(c)}$. □

Exercise 1. Find the theorem in our notes guaranteeing it is enough to prove "Let $y_n \longrightarrow f(c)$ be arbitrary satisfying $\forall n \in \mathbb{N}$, $y_n \neq f(c)$, then

$$\lim_{n\to\infty} \frac{g(y_n) - g(f(c))}{y_n - f(c)} = \frac{1}{f'(c)}.\text{''} \tag{4}$$

Remark 2. The continuity assumption on f cannot be removed.

Problem 1. Find an invertible function $f: [-1, 1] \mapsto [-1, 1]$ with $f(0) = 0$ such that f is differentiable at 0 but its inverse function g is not continuous at 0.

- Calculation of derivative of inverse functions.

Example 3. Calculate $(\ln x)'$.
Solution. Let $y(x) := \ln x$. Then $x = e^{y(x)}$. Taking derivative on both sides, we have

$$1 = e^{y(x)}\, y'(x) \Longrightarrow y'(x) = \frac{1}{e^{y(x)}} = \frac{1}{x}. \tag{5}$$

Example 4. Calculate $(\arcsin x)'$.
Solution. Let $y(x) := \arcsin x$. Then $x = \sin y(x)$. Taking derivative on both sides we have

$$1 = [\cos y(x)]\, y'(x) \Longrightarrow y'(x) = \frac{1}{\cos y(x)}. \tag{6}$$

As $x = \sin y(x)$, we have $(\cos y(x))^2 + x^2 = 1$. Furthermore as arcsin has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, there holds $\cos y(x) \geqslant 0$ and consequently

$$y'(x) = \frac{1}{\cos y(x)} = \frac{1}{\sqrt{1 - x^2}}. \tag{7}$$

Exercise 2. Calculate $(\arccos x)'$.

Example 5. Calculate $(\tan x)'$.
Solution. We have

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x + \sin x\,(\cos x)'}{(\cos x)^2} = \frac{1}{(\cos x)^2}. \tag{8}$$

Example 6. Calculate $(\arctan x)'$.
Solution. Let $y(x) = \arctan x$. We have $x = \tan y(x)$ and therefore

$$1 = (\tan y(x))' = \frac{1}{[\cos y(x)]^2}\, y'(x). \tag{9}$$

Notice

$$\frac{1}{[\cos y(x)]^2} = 1 + (\tan y(x))^2 = 1 + x^2. \tag{10}$$

Therefore $(\arctan x)' = \frac{1}{1 + x^2}$.

Example 7. Let $f(x) := \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $f(x)$ is differentiable everywhere.

Solution. At $a \neq 0$ we have x^2 and $\frac{1}{x}$ differentiable at a and $\sin x$ differentiable at $1/a$. Therefore $f(x)$ is differentiable at a. At $a = 0$ we apply definition

$$\frac{f(x) - f(0)}{x - 0} = x \sin\frac{1}{x} \tag{11}$$

whose limit is 0 thanks to Squeeze. Therefore $f'(0) = 0$.