MATH 117 FALL 2014 LECTURE 34 (Nov. 5, 2014)

Read: Bowman §4.I, 4.J.

- Derivative as function.

  If we let $A := \{a \in \mathbb{R} | f \text{ is differentiable at } a\}$, then $f' : a \mapsto f'(a)$ can be seen as a function with domain A and range $\mathbb{R}$. Thus we often simply write $f'(x)$, as in $(x^3)' = 3\,x^2$.

- Exponential.

PROPOSITION 1. *The exponential function* e^x *is differentiable everywhere if and only if* $\lim_{x \to 0+} \frac{e^x - 1}{x}$ *exists and is finite. Furthermore if we denote this limit by* c_0*, then* $(e^x)' = c_0\, e^x$*.*

Exercise 1. Prove the above proposition.

PROPOSITION 2. $\lim_{x \to 0+} \frac{e^x - 1}{x} = 1$.

Proof. We first prove this limit exists, then prove it is 1.

- $\lim_{x \to 0+} \frac{e^x - 1}{x}$ exists.

  We will prove the function $\frac{e^x - 1}{x}$ is increasing on $(0, +\infty)$, since obviously $\frac{e^x - 1}{x} > 0$ for all $x > 0$, the limit exists.

  Define the function $\varphi_b(x) := \frac{b^x - 1}{x}$ for arbitrary $b > 1$. It suffices to prove that $\varphi_b(x)$ is increasing on $(0, +\infty)$. We prove this in three steps.

  1. $\varphi_b(x)$ is increasing on $\mathbb{N}$. It suffices to prove
  $$\forall n \in \mathbb{N}, \qquad \varphi_b(n+1) > \varphi_b(n). \tag{1}$$
  We notice, as $b > 1$, $b^n > \frac{b^{n-1} + b^{n-2} + \cdots + b + 1}{n}$. Consequently
  $$\begin{aligned} \frac{b^{n+1} - 1}{b^n - 1} &= \frac{(b-1)\,(b^n + b^{n-1} + \cdots + 1)}{(b-1)\,(b^{n-1} + b^{n-2} + \cdots + 1)} \\ &= \frac{b^n}{b^{n-1} + b^{n-2} + \cdots + 1} + 1 > \frac{n+1}{n} \end{aligned} \tag{2}$$
  from which (1) immediately follows.

  2. $\varphi_b(x)$ is increasing on $\mathbb{Q}^+$ (positive rational numbers).

     Let $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q}^+$, then $\frac{p}{q} > \frac{p'}{q'}$ is equivalent to the existence of $m, n, n' \in \mathbb{N}$ such that $\frac{p}{q} = \frac{n}{m}, \frac{p'}{q'} = \frac{n'}{m}$ and $n > n'$. Thus it suffices to prove $\varphi_b\left(\frac{n}{m}\right) > \varphi_b\left(\frac{n'}{m}\right)$ for every such m, n, n'. But this immediately follows from the previous step and the fact $\varphi_b\left(\frac{n}{m}\right) = \varphi_{b^{1/m}}(n)$.

  3. $\varphi_b(x)$ is increasing on $\mathbb{R}^+$ (positive rational numbers).

     Let $x, x' \in \mathbb{R}^+$ be arbitrary with $x > x'$. Then there are sequences $\{x_n\}$, $\{x_n'\} \subset \mathbb{Q}^+$ such that $\lim_{n \to \infty} x_n = x, \lim_{n \to \infty} x_n' = x'$, and $\forall n \in \mathbb{N}, x_n > x, x_n' < x'$. As $\varphi_b(x)$ is continuous, we have
     $$\varphi_b(x) = \lim_{n \to \infty} \varphi_b(x_n) \geqslant \lim_{n \to \infty} \varphi_b(x_n') = \phi_b(x'). \tag{3}$$

  Therefore $c_0 = \lim_{x \to 0+} \varphi_e(x)$ exists.

- $c_0 = 1$. As the limit $\lim_{x\to 0+} \frac{e^x - 1}{x}$ exists, it equals $\lim_{m\to\infty} \frac{e^{1/m}-1}{1/m}$.
  - $c_0 \geqslant 1$. Recall that $e := \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ and $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is increasing. Thus for every $m \in \mathbb{N}$, we have
$$e > \left(1+\frac{1}{m}\right)^m \tag{4}$$
which is equivalent to
$$\frac{e^{1/m}-1}{1/m} > 1. \tag{5}$$
Taking limit $m\to\infty$ and applying Comparison Theorem we have $c_0 \geqslant 1$.
  - $c_0 \leqslant 1$. Recall that $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1}$ and $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is decreasing. Thus for every $k \in \mathbb{N}$, we have $e < \left(1+\frac{1}{k}\right)^{k+1}$. Thus we have
$$\begin{aligned}
\frac{e^{1/m}-1}{1/m} &< m\left[\left(1+\frac{1}{k}\right)^{\frac{k+1}{m}} - 1\right] &(6)\\
&= m\left[\left(\left(1+\frac{1}{k}\right)^{1/m}\right)^{k+1} - 1\right] &(7)\\
&< m\left[\left(1+\frac{1}{m\,k}\right)^{k+1} - 1\right] &(8)\\
&= m\left[\left(1+\frac{k+1}{m\,k}+\binom{k+1}{2}\left(\frac{1}{m\,k}\right)^2+\cdots+\left(\frac{1}{m\,k}\right)^{k+1}\right)-1\right] &(9)\\
&= \frac{k+1}{k}+\binom{k+1}{2}\frac{1}{k^2}\frac{1}{m}+\cdots+\left(\frac{1}{k}\right)^{k+1}\frac{1}{m^k}. &(10)
\end{aligned}$$
Now taking limit $m\to\infty$ and apply Comparison Theorem, we have
$$c_0 = \lim_{m\to\infty} \frac{e^{1/m}-1}{1/m} \leqslant \lim_{m\to\infty} \left\{\frac{k+1}{k}+\binom{k+1}{2}\frac{1}{k^2}\frac{1}{m}+\cdots+\left(\frac{1}{k}\right)^{k+1}\frac{1}{m^k}\right\} = \frac{k+1}{k}. \tag{11}$$
Now taking limit $k\to\infty$ for $c_0 \leqslant \frac{k+1}{k}$ and apply Comparison Theorem, we have $c_0 \leqslant 1$.

  As $c_0 \geqslant 1$ and $c_0 \leqslant 1$, we have $c_0 = 1$.

Thus ends the proof. □

Exercise 2. Make sure you understand every step of (6 – 10).

Exercise 3. Prove $\lim_{m\to\infty}\left\{\frac{k+1}{k}+\binom{k+1}{2}\frac{1}{k^2}\frac{1}{m}+\cdots+\left(\frac{1}{k}\right)^{k+1}\frac{1}{m^k}\right\} = \frac{k+1}{k}$.

- Chain Rule.

THEOREM 3. *Let f be differentiable at $a \in \mathbb{R}$ and g be differentiable at $f(a)$. Then the composite function $g\circ f$ is differentiable at a and $(g\circ f)'(a) = g'(f(a))\cdot f'(a)$.*

Proof. Define
$$G(y) := \begin{cases} \dfrac{g(y)-g(f(a))}{y-f(a)} & y \neq f(a) \\ g'(f(a)) & y = f(a) \end{cases}. \tag{12}$$

Then we have

$$\forall x \neq a, \qquad \frac{(g\circ f)(x)-(g\circ f)(a)}{x-a} = G(f(x))\cdot\frac{f(x)-f(a)}{x-a}. \tag{13}$$

As G, f are continuous, we have $\lim_{x\to a} G(f(x)) = G(f(a)) = g'(f(a))$, as f is differentiable at a, we have $\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$. Thus

$$\lim_{x\to a}\frac{(g\circ f)(x)-(g\circ f)(a)}{x-a} = \lim_{x\to a}\left[G(f(x))\cdot\frac{f(x)-f(a)}{x-a}\right] = g'(f(a))\cdot f'(a) \tag{14}$$

and the conclusion follows. □

Exercise 4. Prove $G(y)$ is continuous.

Exercise 5. Prove (13).