Math 117 Fall 2014 Lecture 32 (Oct. 31, 2014)

In today's lecture $a, b \in \mathbb{R}$. Some of the results can be generalized to the situation $a = -\infty$ or $b = \infty$, some cannot.

- Continuous invertible functions are monotone.

  THEOREM 1. *Let f be continuous on $[a, b]$ and is one-to-one. Then f is either strictly increasing or strictly decreasing.*

  Proof. It suffices to prove that, under the assumptions above,

  - If $f(a) < f(b)$, then f is strictly increasing;
  - If $f(a) > f(b)$, then f is strictly decreasing.

  The proofs for the two cases are almost identical so we will only prove the first one. The proof is divided into two steps.

  - Step 1. Let $x \in (a, b)$ be arbitrary. Then $f(a) < f(x) < f(b)$.

    Assume the contrary. Then there are two cases: $f(x) < f(a) < f(b)$, $f(a) < f(b) < f(x)$. Note that all inequalities are strict as f is one-to-one. We prove the first case and leave the second one as exercise.

    Let $s \in (f(x), f(a))$. As $f(a) < f(b)$, we see that there holds $s \in (f(x), f(b))$. Application of the Intermediate Value Theorem on $[a, x]$ we see that there is $c_1 \in (a, x)$ such that $f(c_1) = s$. Application of IVT on $[x, b]$ gives the existence of $c_2 \in (x, b)$ such that $f(c_2) = s$. As $c_1 < x < c_2$, $c_1 \neq c_2$. This is contradiction to the assumption that f is one-to-one.

  - Step 2. Let $a < x < y < b$ be arbitrary. From Step 1 we know that $f(x) < f(b)$. Now repeat Step 1 for x, y, b (x as a, y as x, and b as b) we see that $f(x) < f(y) < f(b)$.

  Thus we have proved: For arbitrary $a \leqslant x < y \leqslant b$, $f(x) < f(y)$. Thus f is strictly increasing. □

  Exercise 1. Write detailed proof for the case $f(a) < f(b) < f(x)$ in Step 1 of the above proof.

- Continuity of inverse functions

  THEOREM 2. *Let f be one-to-one on $[a, b]$ and continuous. Then f is invertible on $[a, b]$ with inverse function g defined on a closed interval $[c, d]$. Furthermore g is continuous, and g has the same monotonicity as f.*

  Proof. We first prove the existence of g, and then the monotonicity, and finally the continuity.

  - Existence of g.

    All we need to prove is $f([a, b])$ is a closed interval. From Theorem 1 we know that f is either strictly increasing or strictly decreasing. Wlog we assume it is strictly increasing. We claim $f([a, b]) = [f(a), f(b)]$.

    – $f([a, b]) \subseteq [f(a), f(b)]$. Let $x \in [a, b]$ be arbitrary. From Theorem 1 we see that $x \in [a, b] \Longrightarrow f(a) \leqslant f(x) \leqslant f(b) \Longrightarrow f(x) \in [f(a), f(b)]$.

    – $[f(a), f(b)] \subseteq f([a, b])$. Let $y \in [f(a), f(b)]$. By IVT there is $x \in [a, b]$ such that $f(x) = y$ so $y \in f([a, b])$.

    In the following we denote $f([a, b]) = [c, d]$.

- Monotonicity of g.

  Assume f is strictly increasing (the case f is strictly decreasing is almost identical), we prove g is also strictly increasing. Let $c \leqslant y_1 < y_2 \leqslant d$. Assume $g(y_1) \geqslant g(y_2)$. Then $y_1 = f(g(y_1)) \geqslant f(g(y_2)) = y_2$. Contradiction. Therefore $g(y_1) < g(y_2)$.

- Continuity of g.

  Let $y_0 \in (c, d)$ be arbitrary. We will prove $\lim_{y \to y_0+} g(y) = g(y_0)$ and leave $\lim_{y \to y_0-} g(y) = g(y_0)$ as exercise.

  As $y > y_0 \Longrightarrow g(y) > g(y_0)$ and $g(y)$ is decreasing as y approaches y_0 from the right (some times written as $y \searrow y_0$), the limit $\lim_{y \to y_0+} g(y)$ exists. We denote it by L. Following Comparison Theorem $L \geqslant g(y_0)$.

  Assume $L > g(y_0)$. Then we calculate, using the monotonicity and continuity of f, as well as the fact that g is the inverse function of f,

$$y_0 = f(g(y_0)) < f(L) = f\left(\lim_{y \to y_0+} g(y)\right) = \lim_{y \to y_0+} f(g(y)) = \lim_{y \to y_0+} y = y_0. \tag{1}$$

  Contradiction. Therefore $L = g(y_0)$. □

Exercise 2. Prove $\lim_{y \to y_0-} g(y) = g(y_0)$.

Exercise 3. Prove continuity of g at $y = c$ and $y = d$.

Example 3. $\ln x$, the inverse function of e^x, is strictly increasing and continuous on $(0, +\infty)$.

- Max and Min.

THEOREM 4. *Let $f: [a, b] \mapsto \mathbb{R}$ be continuous. Then*

a) *There are $x_M, x_m \in [a, b]$ such that*

$$\forall x \in [a, b], \qquad f(x_m) \leqslant f(x) \leqslant f(x_M). \tag{2}$$

We call x_M a "maximizer" of f over $[a, b]$ and x_m a "minimizer" of f over $[a, b]$.

b) *$f([a, b]) = [f(x_m), f(x_M)]$. In particular f is bounded.*

Proof. We prove the existence of x_M and leave the remaining of the proof as exercises. Denote $L := \sup_{[a,b]} f := \{y | y \in f([a, b])\}$. Then there are $x_n \in [a, b]$ such that $\lim_{n \to \infty} f(x_n) = L$. Since $\{x_n\} \subset [a, b]$ it is a bounded sequence. Following Bolzano-Weierstrass there is a subsequence $\{x_{n_k}\}$ that converges to some $x_M \in \mathbb{R}$.

As $a \leqslant x_{n_k} \leqslant b$, by Comparison we have $a \leqslant x_M \leqslant b$ that is $x_M \in [a, b]$. Furthermore by continuity of f we have

$$f(x_M) = f\left(\lim_{k \to \infty} x_{n_k}\right) = \lim_{k \to \infty} f(x_{n_k}) = L. \tag{3}$$

Thus ends the proof for existence of the maximizer. □

Exercise 4. Prove the existence of x_m.

Exercise 5. Prove part b).