Math 117 Fall 2014 Homework 7 Solutions

Due Thursday Nov. 6 3pm in Assignment Box

Question 1. (5 pts) *Prove by ε-δ that the Heaviside function* $H(x) := \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$ *is continuous at $a \neq 0$ but discontinuous at $a = 0$.*

Proof. Let $a \neq 0$. For every $\varepsilon > 0$, take $\delta = |a|$. Then for every $|x - a| < \delta$, we see that x and a share the same sign. Therefore $|f(x) - f(a)| = 0 < \varepsilon$ and consequently $H(x)$ is continuous at a.

At 0, we prove the working negation of continuity:

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ \exists |x - a| < \delta, \qquad |f(x) - f(a)| \geqslant \varepsilon. \tag{1}$$

Let $\delta > 0$ be arbitrary. Then there is $x > 0$ such that $|x - 0| < \delta$. For this x we have $|f(x) - f(0)| \geqslant 1$. Thus ends the proof. □

Question 2. (5 pts) *Let* $f(x) := \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$. *Prove that $f(x)$ is continuous at every $a \in \mathbb{R}$.*

Proof. First as e^x is continuous everywhere and $-\frac{1}{x}$ is continuous at every $a \neq 0$, the composite function $e^{-1/x}$ is continuous at every $a \neq 0$.

Now let $a \in \mathbb{R}$ be arbitrary. There are three cases.

1. $a > 0$. For every $\varepsilon > 0$, as $e^{-1/x}$ is continuous at a, there is $\delta_1 > 0$ such that $|x - a| < \delta_1 \Longrightarrow \left|e^{-1/x} - e^{-1/a}\right| < \varepsilon$. Set $\delta = \min \{\delta_1, |a|\}$. Then for every $|x - a| < \delta$, we have $x > 0$ and $|x - a| < \delta_1$. Consequently $|f(x) - f(a)| = \left|e^{-1/x} - e^{-1/a}\right| < \varepsilon$.
2. $a < 0$. For every $\varepsilon > 0$, set $\delta = |a|$. For every $|x - a| < \delta$ we have $x < 0$ and $|f(x) - f(a)| = |0 - 0| = 0 < \varepsilon$.
3. $a = 0$. For every $\varepsilon > 0$, set $\delta = \left(\ln\frac{1}{\varepsilon}\right)^{-1}$. Then for every $|x - a| < \delta$, we have two cases:
   a. $x > 0$. In this case $|f(x) - f(a)| = e^{-1/x} < e^{-1/\delta} = \varepsilon$;
   b. $x < 0$. In this case $|f(x) - f(a)| = |0 - 0| = 0 < \varepsilon$.

Thus ends the proof. □

Question 3. (5 pts) *Prove that the equation $7\,x^6 - 9\,x^5 - 1 = 0$ has at least two real solutions.*

Proof. Denote $f(x) := 7\,x^6 - 9\,x^5 - 1$. Since $f(x)$ is a polynomial, it is continuous everywhere. Now observe $f(0) = -1 < 0,\ f(2) = 159 > 0,\ f(-1) = 1 > 0$.

- Apply IVT on $[0, 2]$, we see there is $c_1 \in (0, 2)$ such that $f(c_1) = 0$;
- Apply IVT on $[-1, 0]$, we see there is $c_2 \in (-1, 0)$ such that $f(c_2) = 0$.

As $c_1 > 0 > c_2$ they are different and consequently we have two real solutions. □

Question 4. (5 pts) *Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be such that for every $s \in \mathbb{R}$, there are **exactly** two solutions to $f(x) = s$. Prove that f is not continuous (we say a function is "continuous" if it is continuous everywhere in its domain).*

Proof. Assume the contrary. Since $0 \in \mathbb{R}$ there are $a < b$ such that $f(a) = f(b) = 0$. By IVT either $f(x) > 0$ for all $x \in (a, b)$ or $f(x) < 0$ for all $x \in (a, b)$. Wlog consider the first case. Denote $M := \sup_{[a,b]} f(x) > 0$. By assumption there are $c < d$ such that $f(c) = f(d) = M$ and at least one of them is in (a, b) due to continuity of f (f attains maximum and minimum). There are two cases.

1. One of c, d is in (a, b). Wlog $a < c < b < d$. Let $s \in (0, M)$. Then we have $f(a) < s < f(c)$; $f(c) > s > f(b)$; $f(b) < s < f(d)$. Applying IVT on $[a, c], [c, b], [b, d]$ we see there are $\xi_1 \in (a, c)$, $\xi_2 \in (c, b)$, $\xi_3 \in (b, d)$ such that $f(\xi_1) = f(\xi_2) = f(\xi_3) = s$. Contradiction.

2. Both $c,\ d \in (a,\ b)$. Since $c,\ d$ are the only places where f equals M, we have $\forall x \in (c,\ d)$, $f(x) < M$. Denote $m := \inf_{[c,d]} f(x) < M$. Then by continuity of f there is at least one point $e \in (c,\ d)$ such that $f(e) = m$. Now take $s \in \mathbb{R}$ such that $\max\{m,\ 0\} < s < M$. Application of IVT on $[a, c], [c, e], [e, d], [d, b]$ leads to four different points where f equals s. Contradiction. □