MATH 117 FALL 2014 LECTURE 29 (OCT. 27, 2014)

- Say $f(x)$ is continuous at $a \in \mathbb{R}$ if and only if
$$\lim_{x \to a} f(x) = f(a). \tag{1}$$
This means
  i. $\lim_{x \to a} f(x)$ exists;
  ii. $\lim_{x \to a} f(x) = f(a)$.
- Therefore $f(x)$ is not continuous at $a \in \mathbb{R}$ means either $\lim_{x \to a} f(x)$ does not exist, or it exists but is different from $f(a)$.
- ε-δ definition for continuity.

DEFINITION 1. *$f(x)$ is continuous at $a \in \mathbb{R}$ if and only if*
$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall |x - a| < \delta, \qquad |f(x) - f(a)| < \varepsilon. \tag{2}$$

Exercise 1. Should we require $0 < |x - a| < \delta$ instead of $|x - a| < \delta$?

- Say $f(x)$ is left (right) continuous at $a \in \mathbb{R}$ if and only if
$$\lim_{x \to a-} f(x) = f(a) \qquad \left(\lim_{x \to a+} f(x) = f(a) \right). \tag{3}$$

Exercise 2. Write down the ε-δ definition for left/right continuity.

- Say $f(x)$ is continuous on $[a, b]$ if and only if
  - $f(x)$ is continuous on (a, b);
  - $f(x)$ is left continuous at b;
  - $f(x)$ is right continuous at a.
- Polynomials are continuous everywhere.
  - Building blocks.

Example 2. Let $c \in \mathbb{R}, a \in \mathbb{R}$. Then
  a) $f(x) \equiv c$ (the constant function) is continuous at a;
  b) $f(x) = x$ is continuous at a.

Proof. We prove by definition.
  a) $\forall \varepsilon > 0$, take $\delta = 2$. Then for every $|x - a| < \delta$, we have $|f(x) - f(a)| = |c - c| = 0 < \varepsilon$.
  b) $\forall \varepsilon > 0$, take $\delta = \varepsilon$. Then for every $|x - a| < \delta$, we have $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$. □

  - Assemblage.

THEOREM 3. *Let $f(x), g(x)$ be continuous at $a \in \mathbb{R}$. Then so are $(f + g)(x), (f - g)(x)$, $(f g)(x)$.*

Proof. We prove the first one and leave the other two as exercises.

By the theorem on limit of sum of functions, the existence of $\lim_{x\to a}(f+g)(x)$ follows from the existence of $\lim_{x\to a}f(x)$ and $\lim_{x\to a}g(x)$, and furthermore the same theorem gives

$$\lim_{x\to a}(f+g)(x)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)=f(a)+g(a)=(f+g)(a). \tag{4}$$

Thus ends the proof. □

Exercise 3. Prove the following by induction:
Let $a\in\mathbb{R}$ and $P(x)$ be a polynomial. Then $P(x)$ is continuous at a.

- Rational functions.

THEOREM 4. *Let $f(x),g(x)$ be continuous at $a\in\mathbb{R}$ and furthermore assume $g(a)\neq 0$. Then $\frac{f}{g}$ is continuous at a.*

Proof. As $g(a)\neq 0$, together with continuity of g we have $\lim_{x\to a}g(x)\neq 0$. Application of the theorem on $\lim_{x\to a}\frac{f}{g}$ immediately gives the result. □

COROLLARY 5. *Let $f(x)=\frac{P(x)}{Q(x)}$ where P, Q are polynomials. Then $f(x)$ is continuous at every $a\in\mathbb{R}$ such that $Q(a)\neq 0$.*

Exercise 4. Find $a\in\mathbb{R}$ and P,Q polynomials such that $Q(a)=0, P(a)\neq 0$ and

- $\lim_{x\to a}\frac{P(x)}{Q(x)}=+\infty$; or
- $\lim_{x\to a}\frac{P(x)}{Q(x)}=-\infty$; or
- $\lim_{x\to a}\frac{P(x)}{Q(x)}$ does not exist.

Justify your claims.

Exercise 5. Let P,Q be polynomials and $a\in\mathbb{R}$. Further assume $P(a)\neq 0, Q(a)=0$. Prove: There is no $L\in\mathbb{R}$ such that $\lim_{x\to a}\frac{P(x)}{Q(x)}=L$.