MATH 117 FALL 2014 LECTURE 28 (OCT. 23, 2014)

Example 1. Prove $\lim_{x\to 0}(1+x)^{1/3}=1$.

Solution 1 (Definition). Let $\varepsilon>0$ be arbitrary. Set $\delta<\min\{1,\varepsilon\}$. Then for every $0<|x-0|<\delta$, we have $1+x\geqslant 1-|x|>1-\delta>0$ and therefore

$$\left|(1+x)^{1/3}-1\right|=\frac{|(1+x)-1|}{\left|(1+x)^{2/3}+(1+x)^{1/3}+1\right|}<\frac{|x|}{1}<\delta<\varepsilon. \tag{1}$$

Solution 2 (Squeeze). For $-1<x<1$ we have

$$1-|x|\leqslant(1-|x|)^{1/3}\leqslant(1+x)^{1/3}\leqslant(1+|x|)^{1/3}\leqslant 1+|x|. \tag{2}$$

As $\lim_{x\to 0}(1-|x|)=\lim_{x\to 0}(1+|x|)=0$, the conclusion follows from Squeeze.

Exercise 1. Prove that for every $x\in\mathbb{R}$, $1-|x|\leqslant 1+x\leqslant 1+|x|$.

Exercise 2. Prove by definition $\lim_{x\to 0}(1-|x|)=\lim_{x\to 0}(1+|x|)=0$.

Exercise 3. We have only proved $1-|x|\leqslant(1+x)^{1/3}\leqslant 1+|x|$ for $-1<x<1$, not for all $x\in\mathbb{R}$. Why could we apply Squeeze? Explain.

Example 2. Let $a_n=(-1)^n-e^{-n}$. Calculate $\sup_{n\in\mathbb{N}}a_n$, $\inf_{n\in\mathbb{N}}a_n$, $\limsup_{n\to\infty}a_n$, $\liminf_{n\to\infty}a_n$. Justify.

Solution.

- $\sup_{n\in\mathbb{N}}a_n=1$.
  - 1 is an upper bound.
    $\forall n\in\mathbb{N}$, we have $a_n=(-1)^n-e^{-n}\leqslant 1-e^{-n}<1$.
  - 1 is the least upper bound.
    $\forall b<1$, take $n\in\mathbb{N}$ such that $n>\frac{1}{2}\ln\frac{1}{1-b}$. Then we have $-2\,n<\ln(1-b)\Longrightarrow e^{-2n}<1-b$. Thus $a_{2n}=(-1)^{2n}-e^{-2n}=1-e^{-2n}>1-(1-b)=b$. Therefore b is not an upper bound.
- $\inf_{n\in\mathbb{N}}a_n=-1-e^{-1}$.
  - $-1-e^{-1}$ is a lower bound.
    $\forall n\in\mathbb{N}$, we have $a_n=(-1)^n-e^{-n}\geqslant -1-e^{-n}\geqslant -1-e^{-1}$.
  - $-1-e^{-1}$ is the greatest lower bound.
    $\forall b>-1-e^{-1}$, we have $a_1=-1-e^{-1}<b$ and thus b is not a lower bound.
- $\limsup_{n\to\infty}a_n$.
  Let $M_n:=\sup_{k\geqslant n}a_k=\sup\{a_n,a_{n+1},\ldots\}$.
  - $M_n\leqslant 1$.
    $\forall k\geqslant n$, we have $a_k=(-1)^k-e^{-k}\leqslant(-1)^k\leqslant 1$.
  - $M_n\geqslant 1-e^{-2n}$. It suffices to prove that there is $k\geqslant n$ such that $a_k\geqslant 1-e^{-2n}$. Taking $k=2\,n$ we see that $a_k=1-e^{-2n}$.

  Therefore $1-e^{-2n}\leqslant M_n\leqslant 1$ and application of Squeeze leads to $\lim_{n\to\infty}M_n=1$. Therefore $\limsup_{n\to\infty}a_n=1$.
- $\liminf_{n\to\infty}a_n$.

Let $m_n := \inf_{k \geqslant n} a_k = \inf \{a_n, a_{n+1}, \ldots\}$.

- $m_n \leqslant -1$.

  It suffices to find $k \geqslant n$ such that $a_k \leqslant -1$. Taking $k = 2n+1$ we have $k \geqslant n$ and $a_k = (-1)^{2n+1} - e^{-(2n+1)} = -1 - e^{-(2n+1)} < -1$.

- $m_n \geqslant -1 - e^{-n}$.

  We show that $-1 - e^{-n}$ is a lower bound for $\{a_n, a_{n+1}, \ldots\}$. Let $k \geqslant n$ be arbitrary. Then $a_k = (-1)^k - e^{-k} \geqslant -1 - e^{-k} \geqslant -1 - e^{-n}$ (note that $k \geqslant n$).

Thus $-1 - e^{-n} \leqslant m_n \leqslant -1$ and application of Squeeze leads to $\lim_{n\to\infty} m_n = -1$ which means $\liminf_{n\to\infty} a_n = -1$.

Exercise 4. Let $r \in \mathbb{R}$ and $a_n = r^n$. Prove the following.

a) If $|r| < 1$ then $\lim_{n\to\infty} a_n = 0$;

b) If $r = 1$ then $\lim_{n\to\infty} a_n = 1$;

c) If $r = -1$ then $\limsup_{n\to\infty} a_n = 1, \liminf_{n\to\infty} a_n = -1$;

d) If $r > 1$ then $\lim_{n\to\infty} a_n = +\infty$;

e) If $r < -1$ then $\limsup_{n\to\infty} a_n = +\infty$, $\liminf_{n\to\infty} a_n = -\infty$.