# MATH 117 FALL 2014 LECTURE 27 (OCT. 22, 2014)

**Reading:**

- When should start writing the proof/solution:
  - After understanding what is going on and why the limit is the given (or guessed) number.
  - One way to check: Can give a quick mental estimate of how small $|a_n - a|$ is for say $n = 10101$, 1010101, etc.

    **Example 1.** Prove $\lim_{n\to\infty} \frac{n-1}{n} = 1$.

    **Proof.** Check readiness: $|a_{10101} - 1| = \frac{1}{10101} \leqslant 10^{-4}$ and $|a_{1010101} - 1| = \frac{1}{1010101} \leqslant 10^{-6}$. So we could start writing the proof. □

    **Example 2.** Prove $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

    **Proof.** Check readiness: $|a_{10101} - 0| = \frac{3 \times 3 \times \cdots \times 3}{10101 \times 10100 \times \cdots \times 1} \leqslant !\#@?$. We need to work a bit more before starting to write the proof. First we know that

    $$\frac{3^n}{n!} = \frac{3}{n} \times \frac{3}{n-1} \times \cdots \times \frac{3}{1}. \tag{1}$$

    Intuitively we think the limit is zero because $\frac{3}{k}$ is small for large $k$'s. Thus let's try to split the large $k$'s and small $k$'s:

    $$\frac{3^n}{n!} = \frac{3}{n} \times \cdots \times \frac{3}{4} \times \frac{3}{3} \times \frac{3}{2} \times \frac{3}{1} < \left(\frac{3}{4}\right)^{n-3} \times \frac{9}{2}. \tag{2}$$

    Now we see that we can estimate things like $|a_{10101} - 0|$ and ready to prove.

    Let $\varepsilon > 0$ be arbitrary. Take $N \in \mathbb{N}$ such that $N > \log_{4/3} \frac{9}{2\,\varepsilon} + 3$. Then for every $n \geqslant N$, we have

    $$\left|\frac{3^n}{n!} - 0\right| \frac{3}{n} \times \cdots \times \frac{3}{4} \times \frac{3}{3} \times \frac{3}{2} \times \frac{3}{1} < \left(\frac{3}{4}\right)^{n-3} \times \frac{9}{2} \leqslant \left(\frac{3}{4}\right)^{N-3} \times \frac{9}{2} < \varepsilon. \tag{3}$$

    □

  **Example 3.** Prove $\lim_{n\to\infty} \frac{n}{(n^3 + 2\,n)^{1/3}} = 1$.

  **Proof.** We try to prove through Squeeze. On one hand clearly $\frac{n}{(n^3 + 2\,n)^{1/3}} < \frac{n}{(n^3)^{1/3}} = 1$. On the other hand, $\forall n \in \mathbb{N}$, $n^3 + 2\,n < n^3 + 3\,n^2 + 3\,n + 1 = (n+1)^3$. Therefore $\frac{n}{(n^3 + 2\,n)^{1/3}} > \frac{n}{n+1}$. As $\lim_{n\to\infty} 1 = \lim_{n\to\infty} \frac{n}{n+1} = 1$ we conclude from Squeeze that $\lim_{n\to\infty} \frac{n}{(n^3 + 2\,n)^{1/3}} = 1$. □

- General procedure of proving $\lim_{x\to a} f(x) = L$.
  - For simplicity of presentation we discuss the case $a, L \in \mathbb{R}$ here.
  - Template:

    Let $\varepsilon > 0$ be arbitrary. Set $\delta < C$. For every $0 < |x - a| < \delta$, (Now simplify $|f(x) - L|$):

    $$|f(x) - L| < \cdots < \cdots < \cdots < A < \cdots < B. \tag{4}$$

Here each $<$ is some algebraic manipulation to simplify $|f(x) - L|$. Before $A$ the goal is to "get rid of" all appearances of $x$, thus $A$ should be a formula involving $\delta$ only. From $A$ to $B$ the goal is to simplify this formula of $\delta$ as much as possible, so that the inequality $B < \varepsilon$ (keep in mind that $B$ is a formula of $\delta$) is easy to solve. Once $B < \varepsilon$ is solved, we have the formula for $C$, which should be a formula involving $\varepsilon$.

**Example 4.** Prove $\lim_{x \to 1} \frac{x^2+1}{x^4+1} = 1$ by definition.

**Proof.** Let $\varepsilon > 0$ be arbitrary. We simplify as follows. For $0 < |x - 1| < \delta$,

$$\begin{aligned} \left| \frac{x^2+1}{x^4+1} - 1 \right| &= \left| \frac{x^2 - x^4}{x^4+1} \right| && (5)\\ &\leqslant |x^2|\,|1 - x^2| && (6)\\ (\text{Set } t = x - 1) \quad &= |1+t|^2\,|t|\,|2+t| && (7)\\ &< \delta\,(1+\delta)^2\,(2+\delta). && (8) \end{aligned}$$

Clearly, finding $\delta$ such that $\delta\,(1+\delta)^2\,(2+\delta)$ is difficult so we try to further simplify. From now on we only consider those $\delta < 1$. For such $\delta$ we have

$$\delta\,(1+\delta)^2\,(2+\delta) < 12\,\delta. \tag{9}$$

Thus for any $\delta < 1$ to satisfy $\delta\,(1+\delta)^2\,(2+\delta) < \varepsilon$, it suffices to require $\delta < \frac{\varepsilon}{12}$. As we are considering only those $\delta < 1$, the final requirement of $\delta$ is

$$\delta < \min\left\{1, \frac{\varepsilon}{12}\right\}. \tag{10}$$

Now we finish the proof.

Let $\varepsilon > 0$ be arbitrary. Take $\delta < \min\left\{1, \frac{\varepsilon}{12}\right\}$. Then for every $0 < |x - 1| < \delta$, we have

$$\begin{aligned} \left| \frac{x^2+1}{x^4+1} - 1 \right| &= \left| \frac{x^2 - x^4}{x^4+1} \right| && (11)\\ &\leqslant |x^2|\,|1 - x^2| && (12)\\ (\text{Set } t = x - 1) \quad &= |1+t|^2\,|t|\,|2+t| && (13)\\ &< \delta\,(1+\delta)^2\,(2+\delta) && (14)\\ &< 12\,\delta < \varepsilon. && (15) \end{aligned}$$

Thus ends the proof. □

- A kind of hard problem.

**Problem 1.** Let $d_n > 0$. Assume $\sum_{n=1}^{\infty} d_n = +\infty$. Assume $\lim_{n \to \infty} a_n = a$ ($a$ could be a real number or $\pm\infty$). Prove or disprove:

$$\lim_{n \to \infty} \frac{d_1\,a_1 + \cdots + d_n\,a_n}{d_1 + \cdots + d_n} = a. \tag{16}$$

Would the conclusion still hold if we instead assume $\sum_{n=1}^{\infty} d_n = D \in \mathbb{R}$?