Math 117 Fall 2014 Lecture 26 (Oct. 20, 2014)

Reading:

- Infinite series:
  - The official name for the sum of a list of infinitely many numbers:

$$a_1 + a_2 + a_3 + \cdots \tag{1}$$

  - To define this sum, we consider the "partial sums":

$$\begin{aligned} s_1 &:= a_1; &(2)\\ s_2 &:= a_1 + a_2; &(3)\\ s_3 &:= a_1 + a_2 + a_3; &(4)\\ &\vdots \\ s_n &:= a_1 + a_2 + \cdots + a_n; &(5)\\ &\vdots \end{aligned}$$

    and identify the sum $\sum_{n=1}^{\infty} a_n$ with the limit $\lim_{n\to\infty} s_n$.

    Definition 1. *$\sum_{n=1}^{\infty} a_n = a \in \mathbb{R}$ if and only if $\lim_{n\to\infty} s_n = s$ where s_n is defined as above.*

    Exercise 1. Define $\sum_{n=1}^{\infty} a_n = \pm\infty$.

- Proving convergence of infinite series.
  - Definition.
    $\sum_{n=1}^{\infty} a_n = s$ if and only if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geqslant N, \qquad \left| \sum_{k=1}^{n} a_k - s \right| < \varepsilon. \tag{6}$$

    Exercise 2. Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. (Hint:[1])

    Exercise 3. Prove that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = +\infty$.

    Exercise 4. Let $r \in (-1, 1)$. Prove that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. (Hint:[2])

    Exercise 5. Why is the restriction $r \in (-1, 1)$ necessary in the above exercise?

  - Cauchy.

    Theorem 2. (Cauchy criterion for series) *$\sum_{n=1}^{\infty} a_n$ converges if and only if*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m > n \geqslant N, \qquad \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon. \tag{7}$$

    Example 3. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

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1. $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.
2. Prove by induction that $1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$.

Proof. Let $\varepsilon > 0$ be arbitrary. Take $N > \frac{3}{\varepsilon}$. Then for every $m > n \geqslant N$, we have (here we consider the case $m - n$ is odd. The case $m - n$ is even can be done similarly)

$$\begin{aligned}
\left|\sum_{k=n+1}^{m} a_k\right| &= \left|\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n+2}}{n+2} + \cdots + \frac{(-1)^m}{m}\right| \\
&= \left|(-1)^{n+1}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + (-1)^{n+3}\left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \cdots + \right. \\
&\quad \left. (-1)^{m-2}\left(\frac{1}{m-2} - \frac{1}{m-1}\right) + (-1)^m \frac{1}{m}\right| \\
&\leqslant \left|\frac{1}{n+1} - \frac{1}{n+2}\right| + \left|\frac{1}{n+3} - \frac{1}{n+4}\right| + \cdots + \frac{1}{m} \\
&< \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \cdots + \\
&\quad \left(\frac{1}{m-2} - \frac{1}{m-1}\right) + \frac{1}{m} \\
&= \frac{1}{n+1} - \frac{1}{m-1} + \frac{1}{m} < \frac{3}{n+1} < \frac{3}{N} < \varepsilon. \qquad (8)
\end{aligned}$$

Therefore the series is Cauchy and converges. ☐

Exercise 6. Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Exercise 7. Find a convergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} |a_n|$ does not converge.

- Monotone.
  - Notice that $\{s_n\}$ is increasing if and only if $\forall n$, $a_n \geqslant 0$; $\{s_n\}$ is decreasing if and only if $\forall n, a_n \leqslant 0$.
  - Therefore we consider non-negative and non-positive series.

THEOREM 4. *Let $\sum_{n=1}^{\infty} a_n$ be a non-negative series, that is $\forall n \in \mathbb{N}$, $a_n \geqslant 0$. Then it converges if and only if it has a finite upper bound.*

Problem 1. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

- Squeeze.

THEOREM 5. *Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ be series satisfying*

i. $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n} b_k \leqslant \sum_{k=1}^{n} a_k \leqslant \sum_{k=1}^{n} c_k$;

ii. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n$.

Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n$.

Remark 6. In practice it is usually difficult to apply Squeeze to series for the following reason. It is often hard to calculate the partial sums, on the other hand it is relatively easy to find $b_n \leqslant a_n \leqslant c_n$. However for such b_n, c_n, we would always have $\sum_{n=1}^{\infty} b_n < \sum_{n=1}^{\infty} c_n$ due to the fact that $\forall n \in \mathbb{N}, c_n - b_n \geqslant 0$, unless $\forall n \in \mathbb{N}, b_n = a_n = c_n$.

Exercise 8. Let $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} a_n > 0$ unless $\forall n, a_n = 0$.