Math 117 Fall 2014 Lecture 25 (Oct. 17, 2014)

Reading:

- Recall: For a bounded sequence $\{a_n\}$,

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left[\sup_{k\geqslant n} a_k\right]; \qquad \liminf_{n\to\infty} a_n := \lim_{n\to\infty}\left[\inf_{k\geqslant n} a_k\right]. \tag{1}$$

Theorem 1. *Let $\{a_n\}, \{b_n\}$ be bounded sequences. Then*

$$\limsup_{n\to\infty}(a_n+b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n. \tag{2}$$

Proof. We prove this with two steps.

i. Let $n\in\mathbb{N}$. Then

$$\sup_{k\geqslant n}(a_n+b_n) \leqslant \sup_{k\geqslant n} a_k + \sup_{k\geqslant n} b_k. \tag{3}$$

To show this it suffices to show $\sup_{k\geqslant n} a_k + \sup_{k\geqslant n} b_k$ is an upper bound of the set $\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\}$. Take an arbitrary a from this set. Then there is $l\geqslant n$ such that $a = a_l + b_l$. As $a_l \in \{a_n, a_{n+1}, \ldots\}$, $a_l \leqslant \sup_{k\geqslant n} a_k = \sup\{a_n, a_{n+1}, \ldots\}$. Similarly $b_l \leqslant \sup_{k\geqslant n} b_k$. Putting things together we have $a \leqslant \sup_{k\geqslant n} a_k + \sup_{k\geqslant n} b_k$.

ii. Now that (3) holds, we take limit $n\to\infty$ and (2) follows from Comparison Theorem. □

Exercise 1. Let $\{a_n\}, \{b_n\}$ be bounded sequences. Then

$$\liminf_{n\to\infty}(a_n+b_n) \geqslant \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n. \tag{4}$$

Find an example of $\{a_n\}, \{b_n\}$ such that strict inequality holds.

Exercise 2. Let $\{a_n\}$ be a bounded sequence. Prove

$$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n). \tag{5}$$

Exercise 3. Let $\{a_n\}, \{b_n\}$ be bounded sequences. Prove or disprove:

$$\limsup_{n\to\infty}(a_n+b_n) \geqslant \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n. \tag{6}$$

Problem 1. Let $\{b_n\}$ be a bounded sequence. Prove or disprove:

$$\lim_{n\to\infty} b_n \text{ exists} \Longleftrightarrow \forall \text{ bounded } \{a_n\},\ \limsup_{n\to\infty}(a_n+b_n) = \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n. \tag{7}$$

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Remark 2. Let $\{a_n\}$ be a bounded sequence. Then $\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n$ can be seen as the "eventual magnitude of oscillation" of the sequence.