MATH 117 FALL 2014 LECTURE 23 (OCT. 15, 2014)

Reading:

- Let $\{a_n\}$ be a sequence. Recall:
  - If $\{a_n\}$ converges, then $\{a_n\}$ is bounded. Note that $\lim_{n\to\infty} a_n = \pm\infty$ are called "$\{a_n\}$ diverges to $+\infty/-\infty$."
  - If $\{a_n\}$ is bounded, then it has a convergent subsequence.

  The second fact is one of the most important in the whole field of analysis. Thus it deserves a name:

  THEOREM 1. (BOLZANO-WEIERSTRASS) *Let $\{a_n\}$ be a sequence. If $\{a_n\}$ is bounded, then it has a convergent subsequence.*

- Let $\{a_n\}$ be a sequence. Define the set of its "accumulation points"

$$A(\{a_n\}) := \{a \in \mathbb{R} | \text{There is a subsequence of } \{a_n\} \text{ converging to } a\}. \tag{1}$$

  Then the question comes: What does A look like?

- Convergent sequences.

  THEOREM 2. *Let $\{a_n\}$ be a bounded sequence. Then it is convergent if and only if $A(\{a_n\})$ consists of exactly one number.*

  Proof. The "only if" part follows immediately from the fact that if $\lim_{n\to\infty} a_n = a$ then everyone of its subsequences also converges to a.

  Now we prove the "if" part. Let $A(\{a_n\}) = \{a\}$ consist of exactly one number. Assume that $\lim_{n\to\infty} a_n = a$ does not hold. This means there is $\varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$ there is $n \geqslant N$ such that $|a_n - a| \geqslant \varepsilon_0$. These a_n's form a subsequence, denote it by $\{a_{n_k}\}$. Now as $\{a_n\}$ is bounded, so is $\{a_{n_k}\}$. By Bolzano-Weierstrass we know that there is a convergent subsequence $\{a_{n_{k_l}}\}$. Let $b := \lim_{l\to\infty} a_{n_{k_l}}$. As $\forall l \in \mathbb{N}$, $|a_{n_{k_l}} - a| \geqslant \varepsilon_0$, we have $b \neq a$. This leads to the following contradiction: $b \in A(\{a_n\}) = \{a\}$ but $b \neq a$. □

  Exercise 1. Prove that a subsequence of a subsequence of $\{a_n\}$ is again a subsequence of $\{a_n\}$.

  Exercise 2. Let $\{a_n\}$ be bounded. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Prove that $\{a_{n_k}\}$ is also bounded.

- Two examples.

  Example 3. $a_n = (-1)^n$.

  We claim that $A(\{a_n\}) = \{1, -1\}$ and prove this as follows.

  - $\{1, -1\} \subseteq A(\{a_n\})$. Set $n_k = 2\,k$. We have $a_{n_k} = 1$ for all $k \in \mathbb{N}$ and therefore $\lim_{k\to\infty} a_{n_k} = 1$ and consequently $1 \in A(\{a_n\})$. Similarly we can prove $-1 \in A(\{a_n\})$. Thus $\{1, -1\} \subseteq A(\{a_n\})$.
  - $A(\{a_n\}) \subseteq \{1, -1\}$. Let $b \neq 1, -1$ be arbitrary, we prove $b \notin A(\{a_n\})$. More specifically, let $\{a_{n_k}\}$ be an arbitrary subsequence, we prove $\lim_{k\to\infty} a_{n_k} = b$ cannot hold.

    Let $\varepsilon_0 = \min\{|b-1|, |b+1|\}$. Let $K \in \mathbb{N}$ be arbitrary. Take a $k > K$. We have $a_{n_k} =$ either 1 or -1. Therefore

$$|a_{n_k} - b| \geqslant \varepsilon_0. \tag{2}$$

    So $\lim_{k\to\infty} a_{n_k} = b$ cannot hold.

Exercise 3. Recall the working negation of "$\lim_{n\to\infty} a_n = a$" and see how it is used in the above proof.

Exercise 4. Let $a_n = (-1)^n + e^{-n}$. Find $A(\{a_n\})$ and justify.

Example 4. $a_n = \{n\sqrt{2}\}$ where $\{x\}$ is the fractional part of x, that is $x = \{x\} + n$ where n is the largest integer no bigger than x. For example $\left\{\frac{3}{2}\right\} = \frac{1}{2}$, $\{\sqrt{2}\} = \sqrt{2} - 1$.

We claim that $A(\{a_n\}) = [0, 1]$. The proof is divided into the following steps.

1. $\forall n \in \mathbb{N}$, $a_n > 0$. Assume there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$. Then we have $m \in \mathbb{N}$ such that $n_0\sqrt{2} = m \Longrightarrow \sqrt{2} \in \mathbb{Q}$. Contradiction.
2. Let $b \in [0, 1]$, $b \neq 0$ be arbitrary. Assume there is $a_{n_k} \longrightarrow 0$, then there is $a_{m_k} \longrightarrow b$. We prove this by the following construction: Let $m_1 = 1$. For each $l \in \mathbb{N}$, $l > 1$, we find $m_l > m_{l-1}$ such that $|a_{m_l} - b| < \frac{1}{l}$.

   First note that $|a_1 - b| < \frac{1}{1}$ is satisfied as $a_1 \neq 0$. Now we find a_{m_l} successively as follows. Assume $m_1 < m_2 < \cdots < m_{l-1}$ has already been found.

   Since $a_{n_k} \longrightarrow 0$, there is $k_0 \in \mathbb{N}$, $k_0 > m_{l-1}$ such that $a_{n_{k_0}} < \frac{1}{l}$. Now define
   $$r_0 := \max\left\{r \in \mathbb{N} \,|\, r\, a_{n_{k_0}} < 1\right\}. \tag{3}$$
   Consider the points $a_{n_{k_0}}, 2\,a_{n_{k_0}}, \ldots, r_0\, a_{n_{k_0}}$. They divide $[0, 1]$ into $r_0 + 1$ intervals with each interval shorter than $\frac{1}{l}$. As b must belong to one of them, there is $s_0 \in \{1, 2, \ldots, r_0\}$ such that
   $$\left|s_0\, a_{n_{k_0}} - b\right| < \frac{1}{l}. \tag{4}$$
   Now we take $m_l := s_0\, n_{k_0}$. As $s_0\, a_{n_{k_0}} \in (0, 1)$ there holds $a_{m_l} = a_{s_0 n_{k_0}} = s_0\, a_{n_{k_0}}$ and the proof ends.
3. Now we prove there is $a_{n_k} \longrightarrow 0$. This is equivalent to the claim $\inf_{n\in\mathbb{N}} a_n = 0$. Assume the contrary. Then $\delta_0 := \inf_{n\in\mathbb{N}} \{a_n\} > 0$. By definition of inf, there is $n_0 \in \mathbb{N}$ such that $a_{n_0} \in [\delta_0, 2\,\delta_0)$. Now let
   $$k_0 := \max\{k \in \mathbb{N} \,|\, k\, a_{n_0} < 1\}. \tag{5}$$
   We see that there must hold $k_0\, a_{n_0} < 1 < (k_0 + 1)\, a_{n_0}$. As $(k_0 + 1)\, a_{n_0} - k_0\, a_{n_0} < 2\,\delta_0$ either $(k_0 + 1)\, a_{n_0} - 1 < \delta_0$ or $1 - k_0\, a_{n_0} < \delta_0$.
   - Case $(k_0 + 1)\, a_{n_0} - 1 < \delta_0$. We have $a_{(k_0+1) n_0} = (k_0 + 1)\, a_{n_0} - 1 < \delta_0 = \inf_{n\in\mathbb{N}} a_n$. Contradiction.
   - Case $1 - k_0\, a_{n_0} < \delta_0$. In this case let $r_0 := \max\{r \in \mathbb{N} \,|\, r\,(1 - k_0\, a_{n_0}) < 1\}$. Then we have
     $$r_0\,(1 - k_0\, a_{n_0}) < 1 < (r_0 + 1)\,(1 - k_0\, a_{n_0}) \tag{6}$$
     which gives
     $$r_0 - 1 < r_0\, k_0\, a_{n_0} < (r_0 - 1) + (1 - k_0\, a_{n_0}) < (r_0 - 1) + \delta_0 \tag{7}$$
     which in turn gives
     $$a_{r_0 k_0 n_0} = \{r_0\, k_0\, a_{n_0}\} < \delta_0. \tag{8}$$
     Contradiction again.

Exercise 5. Prove that $\forall n, k \in \mathbb{N}, \{n\, a_k\} = a_{nk}$.

Exercise 6. Prove the equivalence between there is $a_{n_k} \longrightarrow 0$ and $\inf_{n\in\mathbb{N}} a_n = 0$.

Problem 1. Prove that $A(\{\sin n\}) = [-1, 1]$.

Problem 2. Calculate $A(\{e^{-n} + \sin n\})$. Justify.

Problem 3. Calculate $A(\{(-1)^n + \sin n\})$. Justify.