Math 117 Fall 2014 Lecture 21 (Oct. 9, 2014)

Reading: Bowman §2.E.

- Note that "convergence" means converging to a number. It does not include the cases $a_n \to +\infty$ and $a_n \to -\infty$. Please make sure you think about whether the following results still apply to these two cases.
- Let $\{a_n\}$ be a sequence. Recall so far we have two methods to prove convergence:
  i. By definition;
  ii. By the following facts:
     - If $\{a_n\}$ is increasing and has an upper bound, then it converges;
     - If $\{a_n\}$ is decreasing and has a lower bound, then it convergens.

  In light of

  Definition 1. *A sequence is said to be monotone if it is increasing or it is decreasing. A sequence is said to be bounded if it has both upper bound and lower bound.*

  we have
     - If $\{a_n\}$ is monotone and bounded, then it converges.

  Exercise 1. Prove the above claim.

  Note that there is a difference between i and ii. On one hand i requires us to know the limit $a \in \mathbb{R}$ while ii does not; On the other hand successful application of i gives us the limit while successful application of ii does not give us this information – we know $\{a_n\}$ converges, but still do not know what its limit is.
- In the following we introduce two other methods that can help us proving convergence.
- Squeeze Theorem.

Theorem 2. (Squeeze, Sandwich, etc.) *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences. If*

i. $\forall n \in \mathbb{N}, a_n \leqslant b_n \leqslant c_n$, and

ii. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,

then $\{b_n\}$ also converges and its limit is L too.

Exercise 2. Prove the theorem. (Hint: Should you use definition, or the facts about monotone sequences?)

Exercise 3. Prove the following: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Exercise 4. Prove or disprove: If $\lim_{n\to\infty} |a_n|$ exists, so does $\lim_{n\to\infty} a_n$.

Exercise 5. State and prove the "function limit" version of Squeeze Theorem.

Example 3. Since $\lim_{n\to\infty} n^{-1} = 0$ and $-n^{-1} \leqslant \frac{\sin n^2}{n} \leqslant n^{-1}$, we have $\lim_{n\to\infty} \frac{\sin n^2}{n} = 0$.

Remark 4. Please note that there are two conclusions in Squeeze Theorem: 1. $\{b_n\}$ converges, that is $\lim_{n\to\infty} b_n$ exists; 2. This limit is the same number as $\lim_{n\to\infty} a_n$ or $\lim_{n\to\infty} c_n$.

Exercise 6. Explain the difference between Squeeze Theorem and Comparison Theorem.

- Cauchy criterion.

DEFINITION 5. (CAUCHY) *A sequence $\{a_n\}$ is said to be a Cauchy sequence (or simply Cauchy) if and only if*

$$\forall \varepsilon > 0 \ \exists N > 0 \ \forall m > n \geqslant N, \qquad |a_m - a_n| < \varepsilon. \tag{1}$$

THEOREM 6. *A sequence $\{a_n\}$ is convergent if and only if $\{a_n\}$ is Cauchy.*

Proof. We prove the easier "only if" first.

- Only if. We need to show that if $\{a_n\}$ is convergent then it is Cauchy. Let $\varepsilon > 0$ be arbitrary. As $\{a_n\}$ is convergent, there is a limit a, and furthermore there is $N \in \mathbb{N}$ such that $\forall n \geqslant N$, $|a_n - a| < \frac{\varepsilon}{2}$. Now for every $m > n \geqslant N$, we have

$$|a_m - a_n| = |(a_m - a) - (a_n - a)| \leqslant |a_m - a| + |a_n - a| < \varepsilon. \tag{2}$$

  Therefore $\{a_n\}$ is Cauchy.

- If. We need to show that if $\{a_n\}$ is Cauchy then it is convergent.

  Note 7. Recall that so far we have three methods to prove convergence: definition; squeeze; monotone. To prove by definition we need to know the limit – not likely here. It turns out either squeeze or monotone would work. Here we will try to create a monotone "subsequence" that is bounded.

  LEMMA 8. *If $\{a_n\}$ is Cauchy then it is bounded.*

  Exercise 7. Prove this lemma. (Hint: Recall how we proved if $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ then $\{a_n\}$ is bounded. Try to follow the same idea.)

  DEFINITION 9. (SUBSEQUENCE) *Let $\{a_n\}$ be a sequence. Then another sequence $\{b_k\}$ is said to be a subsequence of $\{a_n\}$ if and only if*

  i. for every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $b_k = a_{n_k}$;

  ii. $n_1 < n_2 < n_3 < \cdots$. [1]

  Exercise 8. Prove that for every $k \in \mathbb{N}$, $n_k \geqslant k$.

  NOTATION 10. *Usually we simply use $\{a_{n_k}\}$ to denote the subsequence. Note that in this notation k is the running index, that is $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$, while n is just there to remind us that this is a subsuquence of $\{a_n\}$.*

  LEMMA 11. *Let $\{a_n\}$ be Cauchy. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Assume that $\lim_{k\to\infty} a_{n_k} = a \in \mathbb{R}$. Then $\lim_{n\to\infty} a_n = a \in \mathbb{R}$.*

  Proof. Let $\varepsilon > 0$ be arbitrary. As $\lim_{k\to\infty} a_{n_k} = a$, there is $K \in \mathbb{N}$ such that $\forall k \geqslant K$, $|a_{n_k} - a| < \frac{\varepsilon}{2}$. On the other hand, as $\{a_n\}$ is Cauchy, there is $N_0 \in \mathbb{N}$ such that $\forall m > n \geqslant N_0$, $|a_m - a_n| < \frac{\varepsilon}{2}$. Then there is $k_0 \in \mathbb{N}$, $k_0 \geqslant K$ such that $n_{k_0} > N_0$.

  Now set $N := n_{k_0}$. For every $n \geqslant N$, we have

$$|a_n - a| = \left|\left(a_n - a_{n_{k_0}}\right) + \left(a_{n_{k_0}} - a\right)\right| < \varepsilon. \tag{3}$$

1. Note that $n_1, n_2, n_3, \ldots$ must be **strictly increasing**.

Thus ends the proof. □

Now we put in the last piece of puzzle.

LEMMA 12. *Let $\{a_n\}$ be bounded. Then it has a convergent monotone subsequence.*

Proof. As $\{a_n\}$ is bounded, so is any of its subsequences. Therefore all we need to do is to prove the existence of a monotone subsequence.

Consider $M_1 := \sup \{a_1, a_2, \ldots\}$. If there is no $n \in \mathbb{N}$ such that $M_1 = a_n$, then there exists a increasing subsequence (see exercise below). Otherwise there is $n_1 \in \mathbb{N}$ such that $a_{n_1} = M_1$.

Now consider $M_2 := \sup \{a_{n_1+1}, a_{n_1+2}, \ldots\}$. Necessarily $M_2 \leqslant M_1$. If there is no $n \geqslant n_1 + 1$ such that $a_n = M_2$, then we have a increasing sequence again. Otherwise there is $n_2 > n_1$ such that $a_{n_2} = M_2 \leqslant M_1 = a_{n_1}$.

We repeat this process. If at any step the supreme is not reached, we have a increasing subsequence and the process terminates. If this process goes on forever, we have a decreasing subsequence. □

Exercise 9. Apply the above process to the sequence $\{\cos n\}$ to find one increasing subsequence and one decreasing subsequence. For each write down the first five terms. You may need a computer to do this.

Putting the above lemma together we have proved the theorem. □

Exercise 10. Let $M := \sup \{a_1, \ldots\}$. Prove that, if there is no $n \in \mathbb{N}$ such that $M = a_n$, then there exists a increasing subsequence $\{a_{n_k}\}$ such that $\lim_{k \to \infty} a_{n_k} = M$. (Hint:[2])

Problem 1. Try to prove the "if" part through application of Squeeze Theorem. (Hint: Check out "Nested Intervals" in the textbooks or any proof-based introductory calculus book)

Remark 13. The "If" part is called the "Cauchy criterion".

Exercise 11. Can we use the Cauchy criterion to conclude $\lim_{n \to \infty} a_n = +\infty$?

- Not Cauchy.
  - The working negation of Cauchy:

$$\exists \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists m > n \geqslant N, \qquad |a_m - a_n| \geqslant \varepsilon. \tag{4}$$

    This is one of the best methods to prove the non-existence of limit.

    Exercise 12. Let $\{a_n\}$ be a sequence and $\{a^1_{n_k}\}$ and $\{a^2_{n_k}\}$ be two subsequences. Further assume that $\lim_{k \to \infty} a^1_{n_k} = a^1 \neq a^2 = \lim_{k \to \infty} a^2_{n_k}$. Prove that $\lim_{n \to \infty} a_n$ does not exist. Further prove that $\lim_{n \to \infty} a_n = +\infty$ or $-\infty$ also cannot hold.

  - For example let's prove that $\{H_n\}$ where

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \tag{5}$$

    is not Cauchy.

    Proof. Take $\varepsilon = \frac{1}{2}$. Let $N \in \mathbb{N}$, set $m = 2N, n = N + 1$. Then we have $m > n \geqslant N$ and

$$|H_m - H_n| = \frac{1}{N+1} + \cdots + \frac{1}{2N} > \frac{N}{2N} = \frac{1}{2}. \tag{6}$$

2. Take $n_1 = 1$. Then as $a_1 < M_0$ cannot be an upper bound of $\{a_1, a_2, \ldots\}$, there is $n_2 > 1$ such that $a_{n_2} > a_{n_1} = a_1$. Now try to repeat this argument to find $n_3, n_4, n_5, \ldots$ one by one. Note that we have two requirements to meet: 1. $n_1 < n_2 < n_3 < \cdots$; 2. $a_{n_1} < a_{n_2} < a_{n_3} < \cdots$.

Therefore $\{H_n\}$ is not Cauchy. $\square$

Exercise 13. Can we conclude from this that $\lim_{n\to\infty} H_n = +\infty$?

Exercise 14. Prove that $\{\ln n\}$ is not convergent.