Math 117 Fall 2014 Lecture 15 (Sept. 29, 2014)

Reading: Bowman: §2.A, §2.B.

- Sequence.

  Definition 1. *A sequence of real numbers is a function* $a: \mathbb{N} \mapsto \mathbb{R}$.

  We often write $a(n)$ as a_n and denote the whole sequence by $\{a_n\}$ or $a_1, a_2, a_3, \ldots$

  Note. The difference between the sequence $a_1, a_2, a_3, \ldots$ and the set $\{a_1, a_2, a_3, \ldots\}$ is that in a sequence the order matters. $\{a_1, a_2, a_3, \ldots\}$ and $\{a_2, a_1, a_3, \ldots\}$ are the same set but $a_1, a_2, a_3, \ldots$ and $a_2, a_1, a_3, \ldots$ are not the same sequence.

- Limits of increasing and decreasing sequences.

  Lemma 2. *Let $\{a_n\}$ be a increasing sequence. Let $A := \{a_1, a_2, a_3, \ldots\}$ be the set consisting of all the a_n's. Let $a \in \mathbb{R}$ be an upper bound of A. Then $a = \sup A$ if and only if*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geqslant N \qquad a - a_n < \varepsilon. \tag{1}$$

  Remark 3. We see that it is reasonable to call $\sup A$ the "limit" of $\{a_n\}$.

  Proof. Since we need to prove "if and only if", there are two steps.

  - If. We need to show (1) implies $a = \sup A$.

    Let $b < a$ be arbitrary. Set $\varepsilon = a - b$. Then there is $N \in \mathbb{N}$ such that for every $n \geqslant N$, $a - a_n < a - b$ which means $a_n > b$. As all these $a_n \in A$, b is not a upper bound of A. Therefore a is the lowest upper bound, that is $\sup A$.

  - Only if. We need to show $a = \sup A$ implies (1).

    Let $\varepsilon > 0$ be arbitrary. Set $b = a - \varepsilon$. As $b < a = \sup A$ it is not a upper bound of A. Therefore there is $N \in \mathbb{N}$ such that $a_N > b$. This gives

$$a - a_N < a - b = \varepsilon. \tag{2}$$

    Now as $\{a_n\}$ is increasing, we have for every $n \geqslant N$, $a_n \geqslant a_N$ which gives $a - a_n \leqslant a - a_N < \varepsilon$. Therefore (1) holds. □

  Exercise 1. Formulate and prove a similar result for decreasing sequences.

- Limit for a sequence.

  Definition 4. *Let $\{a_n\}$ be a sequence of real numbers. A number $a \in \mathbb{R}$ is said to be the limit of $\{a_n\}$, denoted $a = \lim_{n \to \infty} a_n$, if and only if*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geqslant N, \qquad |a_n - a| < \varepsilon. \tag{3}$$

  Example 5. Find the limit of $\left\{\frac{(-1)^n}{n}\right\}$. Justify.

  Solution. We guess $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$. Now we prove this.

  Let $\varepsilon > 0$ be arbitrary. Set $N > \varepsilon^{-1}$. Then we have for all $n \geqslant N$

$$\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} \leqslant \frac{1}{N} < \frac{1}{\varepsilon^{-1}} = \varepsilon. \tag{4}$$

Thus ends the proof.

Remark 6. It is easy to see that the choice $N > \varepsilon^{-1}$ can only be made after most of the calculations in (4) is done. Only when we reach $\left|\frac{(-1)^n}{n} - 0\right| \leqslant \frac{1}{N}$ has it become clear that the choice of N should be such that $\frac{1}{N} < \varepsilon$.

Example 7. Prove

$$\lim_{n \to \infty} \frac{\sin n^2}{\sqrt{n}} = 0. \tag{5}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Set $N > \varepsilon^{-2}$. Then for every $n \geqslant N$,

$$\left|\frac{\sin n^2}{\sqrt{n}} - 0\right| = \frac{|\sin n^2|}{\sqrt{n}} \leqslant \frac{1}{\sqrt{n}} \leqslant \frac{1}{\sqrt{N}} < \varepsilon. \tag{6}$$

The proof ends. □

Exercise 2. Prove

$$\lim_{n \to \infty} e^{-n} = 0. \tag{7}$$

by definition.

Exercise 3. Let $\{a_n\}$ be a sequence satisfying $\lim_{n \to \infty} a_n = 0$. Prove by definition

$$\lim_{n \to \infty} a_n^2 = 0. \tag{8}$$

Exercise 4. Let $\{a_n\}$ be a sequence satisfying $\lim_{n \to \infty} a_n = 0$. Prove by definition

$$\lim_{n \to \infty} \sqrt{|a_n|} = 0. \tag{9}$$