Math 117 Fall 2014 Midterm Exam 1 Solutions

Sept. 26, 2014 10am - 10:50am. Total 20+2 Pts

NAME: ID#:

- There are five questions.
- Please write clearly and show enough work.

Question 1. (5 pts) *Prove that*

a) **(2 pts)** 19 *is prime.*

b) **(3 pts)** $\sqrt{19}$ *is irrational.*

Proof.

a) First we know that any $b > 19$ cannot divide 19. Next we check: $2\nmid 19$, $3\nmid 19, \ldots, 18\nmid 19$. Thus 19 is prime.

b) Assume the contrary. Then there are $p, q \in \mathbb{Z}, q > 0, (p, q) = 1$ such that $\sqrt{19} = \frac{p}{q}$. This gives $p^2 = 19\, q^2$. Thus $19 | p^2$. By the corollary of the fundamental theorem of arithmetic, $19 | p$. Therefore there is $k \in \mathbb{Z}$ such that $p = 19\, k$. Substituting back we have $(19\, k)^2 = 19\, q^2$ which leads to $q^2 = 19\, k^2$. By a similar argument as before we see that $19 | q$. This is a contradiction to $(p, q) = 1$. □

Comments.

- Most of you did quite well on this one. The only issue is that some of you from $p^2 = 19\, q^2$ directly jump to $p = 19\, k$ or $19 | p$. The steps you skipped are arguably the most crucial step of the whole proof.

Question 2. (5 pts) *Let* $A := \left\{ \frac{1}{m^2 + 1} | m \in \mathbb{Z} \right\}$. *Calculate* $\inf A$. *Justify.*

Solution. First we guess $\inf A = 0$. To justify we prove

- 0 is a lower bound. As $m^2 + 1 \geqslant 1 > 0$, we have $\frac{1}{m^2 + 1} > 0$ for every $m \in \mathbb{Z}$. Therefore 0 is a lower bound.

- 0 is the greatest lower bound. Take an arbitrary $b>0$. There are two cases.[1] If $b \geqslant 1$, we have

$$\frac{1}{1^2+1} = \frac{1}{2} < 1 \leqslant b \tag{1}$$

therefore b cannot be a lower bound. If $b<1$ we take $m \in \mathbb{Z}$ such that $m > \sqrt{\frac{1}{b} - 1}$. Then we have $m^2+1 > \frac{1}{b}$ which leads to $\frac{1}{m^2+1} < b$. Therefore again b cannot be a lower bound. Summarizing, we see that no $b>0$ could be a lower bound and therefore 0 is the greatest lower bound.

Comments.

- The biggest issue for this problem is that most of you did not start from definition of $\inf A$, but instead try to apply your own, imprecise, idea of what $\inf A$ is. The problem with doing this is that you stay at about the same level of rigor as Newton or Archimedes. As a consequence most of you would not be able to go any further in calculus/analysis than these two old men.
- Many of you use arguments like $\frac{1}{m^2+1} \longrightarrow \frac{1}{\infty^2+1} = \frac{1}{\infty} = 0$. This is good intuition but incorrect proof. Furthermore, notice that if you start from the definition of inf, then you do not need to invoke the idea of limit at all.
- Some of you confuse $\inf A$ with $\min A$. Note that the difference is that $\min A \in A$ while $\inf A$ may or may not be a member of A. As a consequence, $\min A$ may not exist but $\inf A$ always exists.
- Some of you write things like

  $$\text{there is } m \in \mathbb{N} \text{ such that } m > \sqrt{\frac{1}{b} - 1} \text{ for every } b > 0.$$

  Note that there is no such m. “for every $b>0$” should be put before “there is $m \in \mathbb{N}$”.
- Also quite a few only proved $0 < \frac{1}{m^2+1}$ for all m and then claim 0 is the infimum.

Question 3. (5 pts) *Prove that the sequence*

$$\sqrt{2}, \sqrt{2\sqrt{3}}, \sqrt{2\sqrt{3\sqrt{2}}}, \sqrt{2\sqrt{3\sqrt{2\sqrt{3}}}}, \ldots \tag{2}$$

1. Note that as b is “arbitrary”, it may or may not be less than 1.

is increasing and has an upper bound. Then find its limit.

Proof. We denote the terms in the sequence by a_n.

- $\{a_n\}$ is increasing.

  We notice that, for every $n \in \mathbb{N}$,

  $$a_{2n+2} = \sqrt{2\sqrt{3\,a_{2n}}}, \qquad a_{2n+1} = \sqrt{2\sqrt{3\,a_{2n-1}}}. \tag{3}$$

  Now we prove by induction that $a_{2n-1} < a_{2n} < a_{2n+1}$ for every $n \in \mathbb{N}$. Note that this implies $\{a_n\}$ is increasing.

  - $n=1$. We have $a_1 = \sqrt{2} = \sqrt{2\sqrt{1}} < \sqrt{2\sqrt{3}} = a_2 = \sqrt{2\sqrt{3\sqrt{1}}} < \sqrt{2\sqrt{3\sqrt{2}}} = a_3$.
  - Assume $a_{2k-1} < a_{2k} < a_{2k+1}$. Then we have

    $$a_{2(k+1)-1} = a_{2k+1} = \sqrt{2\sqrt{3\,a_{2k-1}}} < \sqrt{2\sqrt{3\,a_{2k}}} = a_{2(k+1)} \tag{4}$$

    and similarly $a_{2k+2} < a_{2k+3}$.

  Therefore $\{a_n\}$ is increasing.

- $\{a_n\}$ has an upper bound.

  We prove by induction that $a_{2n}, a_{2n-1} < 3$ for all $n \in \mathbb{N}$.

  - $n=1$. We have $a_1 = \sqrt{2} < 3, a_2 = \sqrt{2\sqrt{3}} < \sqrt{2\sqrt{4}} = 2 < 3$.
  - Assume $a_{2k}, a_{2k-1} < 3$. We have

    $$a_{2k+2} = \sqrt{2\sqrt{3\,a_{2k}}} < \sqrt{2\sqrt{3\times 3}} = \sqrt{6} < 3 \tag{5}$$

    and

    $$a_{2(k+1)-1} = a_{2k+1} = \sqrt{2\sqrt{3\,a_{2k-1}}} < \sqrt{2\sqrt{3\times 3}} = \sqrt{6} < 3. \tag{6}$$

  Thus the upper bound is proved.

- The limit.

  As $\{a_n\}$ is increasing and with an upper bound, it has a limit. Denote this limit by x. Then since

  $$a_n = \sqrt{2\sqrt{3\,a_{n-2}}}, \tag{7}$$

  taking limits on both sides we have $x = \sqrt{2\sqrt{3\,x}}$ which gives $x^4 - 12\,x = 0$ and consequently $x = 0$ or $x = 12^{1/3}$. Now as $a_1 = \sqrt{2} > 0$ and $a_n \geqslant a_1$ for every $n \in \mathbb{N}$, the limit could not be 0. Therefore the limit is $12^{1/3}$. □

Comments.

- One common mistake is not getting the correct relation (recursive formula) between the a_n's.
- Another is that many "proved" $a_n < 2$ which could not be correct as $a_3 = \sqrt{2\sqrt{3\sqrt{2}}} > 2$. Before you prove of yours conjectures, it is important to "test" with a few examples to make sure your guess is correct.
- Some of you wrote a succession of formulas without any indication of the logical relations between them. It is important to clearly state such relations: Does the first formula imply the second? Or the second imply the first? Or they are equivalent?

Question 4. (5 pts) *Let A_n be a sequence of sets. Its "limit supreme" is defined as the set*

$$\limsup_{n\to\infty} A_n := \cap_{n\in\mathbb{N}} \cup_{k\in\mathbb{N},k\geqslant n} A_k. \tag{8}$$

Here $\cup_{k\in\mathbb{N},k\geqslant n} A_k$ *means* $A_n \cup A_{n+1} \cup A_{n+2} \cup \cdots$. *Let* $A_n := [0, 2 + (-1)^n]$. *Calculate* $\limsup_{n\to\infty} A_n$. *Justify your answer.*

Solution.

We first observe that $A_n = [0, 1]$ when n is odd and $[0, 3]$ when n is even. Now we first prove

$$\cup_{k\in\mathbb{N},k\geqslant n} A_k = [0, 3] \tag{9}$$

for every $n \in \mathbb{N}$.

- $[0, 3] \subseteq \cup_{k\in\mathbb{N},k\geqslant n} A_k$. Take an arbitrary $x \in [0, 3]$. Now set $k = 2\,n \geqslant n$. We have $A_{2n} = [0, 3]$ which means $x \in A_{2n}$. Thus by definition of union of sets we see that $[0, 3] \subseteq \cup_{k\in\mathbb{N},k\geqslant n} A_k$.
- $\cup_{k\in\mathbb{N},k\geqslant n} A_k \subseteq [0, 3]$. Let $x \in \cup_{k\in\mathbb{N},k\geqslant n} A_k$ be arbitrary. By definition of the union, there is $k \geqslant n$ such that $x \in A_k$. Two cases.
    1. k is even. Then $x \in A_k = [0, 3]$.
    2. k is odd. Then $x \in A_k = [0, 1]$. As $[0, 1] \subseteq [0, 3]$ we see that $x \in [0, 3]$.

Now we prove

$$\limsup_{n\to\infty} A_n = [0, 3] \cap [0, 3] \cap [0, 3] \cap \cdots = [0, 3]. \tag{10}$$

- $[0, 3] \cap [0, 3] \cap \cdots \subseteq [0, 3]$.

Take $x \in [0,\ 3] \cap [0,\ 3] \cap [0,\ 3] \cap \cdots$ arbitrarily. By definition of intersection of sets we have $x \in [0, 3]$.

- $[0, 3] \subseteq [0, 3] \cap [0, 3] \cap \cdots$.

  Take $x \in [0, 3]$ arbitrarily. Then x is a member of every $[0, 3]$ in the intersection and therefore x belongs to the intersection.

Comments.

- The goal of this problem is to test your ability to work with new definitions. This ability is important in studying modern mathematics.
- Some of you confuse limsup of **a sequence of sets** (as defined here in the Question) with sup of **one single set** and give the number 3 as the answer. However there should not be any confusion if you "start from definition", as limsup is precisely defined in the problem and therefore it does not matter what $\sup A$ means.
- Some of you confuse the interval $[1, 3]$ with the set $\{1, 3\}$. Note that the former is defined as $\{x | 1 \leqslant x \leqslant 3\}$.

Question 5. (Extra 2 pts) *Prove or disprove the following claim:*

$$\sqrt{n\,(n+p^2)} \text{ is irrational for every } n \in \mathbb{N} \text{ and every } p \text{ prime.}$$

Solution.

The claim is false as we can take $n=3, p=3$ to reach

$$\sqrt{n\,(n+p^2)} = \sqrt{3 \times (3+9)} = 6 \in \mathbb{Q}. \qquad (11)$$

Comments.

- The original idea is to see whether you realize the fact (which is an exercise in one of the lecture notes) that if $m \in \mathbb{N}$, then $\sqrt{m} \in \mathbb{Q}$ if and only if m is a square. Therefore we could try setting $n = m^2$ and simplify the formula as

  $$\sqrt{n\,(n+p^2)} = \sqrt{m^2\,(m^2+p^2)} = m\,\sqrt{m^2+p^2}. \qquad (12)$$

  Thus all we need is $m^2 + p^2 = k^2$ for some k. Now remember the classical $(3, 4, 5)$. I didn't realize $n = 3$ would work though.