Math 117 Fall 2014 Homework 3 Solutions

Due Thursday Oct. 2 3pm in Assignment Box

Question 1. (5 pts) *Let $B \subseteq Y$ and $f: X \mapsto Y$. Prove that $f(f^{-1}(B)) \subseteq B$. Can we replace $\subseteq$ by $=$? Justify your claim.*

Proof.

- $f(f^{-1}(B)) \subseteq B$. Take an arbitrary $y \in f(f^{-1}(B))$. By definition of image there is $x \in f^{-1}(B)$ such that $y = f(x)$. Now as $x \in f^{-1}(B)$, by definition of pre-image $f(x) \in B$. Therefore $y \in f(B)$.
- No. For example take $f: \mathbb{R} \mapsto \mathbb{R}$ to be such that $f(x) = 1$ for all $x \in \mathbb{R}$. Now take $B = \{1, 2\}$. Then $f^{-1}(B) = \mathbb{R}$ and $f(f^{-1}(B)) = \{1\} \neq \{1, 2\}$. $\square$

Question 2. (5 pts) *Prove*

$$\binom{n+d-2}{d-2} + \binom{n+d-3}{d-2} = \binom{n+d-1}{n} - \binom{n+d-3}{n-2} \tag{1}$$

for every pair of natural numbers $n, d \geqslant 2$.

Proof. We use the identities

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \qquad \text{and} \qquad \binom{n}{k} = \binom{n}{n-k}. \tag{2}$$

These leads to

$$\begin{aligned} \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2} + \binom{n+d-3}{n-2} &= \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2} + \binom{n+d-3}{d-1} \\ &= \binom{n+d-2}{d-2} + \binom{n+d-2}{d-1} \\ &= \binom{n+d-1}{d-1} \\ &= \binom{n+d-1}{n}. \end{aligned} \tag{3}$$

$\square$

Question 3. (5 pts) *A function $f: \mathbb{R} \mapsto \mathbb{R}$ is said to be "Lipschitz" if the following holds:*

$$\exists M > 0 \quad \forall x, y \in \mathbb{R} \qquad |f(x) - f(y)| \leqslant M\, |x - y|. \tag{4}$$

a) (2 pt) *Find the working negation to (4).*

b) (3 pts) *Is $f(x) = x$ Lipschitz? Is $g(x) = x^2$ Lipschitz? Justify your claims.*

Solution.

a) It is

$$\forall M > 0 \quad \exists x, y \in \mathbb{R} \qquad |f(x) - f(y)| > M\, |x - y|. \tag{5}$$

b)

- $f(x) = x$ is Lipschitz.

Proof. Let $x, y \in \mathbb{R}$ be arbitrary. Then we have

$$|f(x) - f(y)| = |x - y| \leqslant 1 \cdot |x - y|. \tag{6}$$

Therefore $f(x)$ is Lipschitz. □

- $g(x) = x^2$ is not Lipschitz.

Proof. Let $M > 0$ be arbitrary. Set $x = M$, $y = M + 1$. Then we have

$$|f(x) - f(y)| = 2M + 1 > M = M \cdot |x - y|. \tag{7}$$

Therefore $g(x)$ is not Lipschitz. □

QUESTION 4. (5 PTS) *Recall that in the computation of* $\sqrt{2}$*, we have the*

- *Babylonian method:* $a_1 > 0$, $b_1 = 2/a_1$,

$$a_{n+1} = \frac{1}{2}(a_n + b_n), \qquad b_{n+1} = \frac{1}{\frac{1}{2}\left(\frac{1}{a_n} + \frac{1}{b_n}\right)}, \tag{8}$$

- *and Newton's method:* $x_1 > 0$,

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}. \tag{9}$$

Someone in our class made the conjecture that the two methods are identical. Prove or disprove his conjecture.

Proof. Yes the two methods are identical. It suffices to prove that, in the Babylonian method, we have

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}. \tag{10}$$

In the following we prove that for every n, $a_n b_n = 2$. Once this is done, we have

$$a_{n+1} = \frac{1}{2}(a_n + b_n) = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{a_n}{2} + \frac{1}{a_n} \tag{11}$$

as desired.

We prove by induction.

- $n = 1$. Since $b_1 = \frac{2}{a_1}$, we have $a_1 b_1 = 2$.
- Assume that $a_k b_k = 2$. Then we have

$$a_{k+1} b_{k+1} = \frac{a_k + b_k}{2} \cdot \frac{1}{\frac{1}{2}\left(\frac{1}{a_k} + \frac{1}{b_k}\right)} = \frac{a_k + b_k}{2} \cdot \frac{2 a_k b_k}{a_k + b_k} = a_k b_k = 2. \tag{12}$$

Thus the proof ends. □