Math 117 Fall 2014 Lecture 11 (Sept. 19, 2014)

Reading: 314 Notes: Sets and Functions §3.3, §3.5. Dr. Bowman's book: §3.A, 3.B.

- Recall function.
  - A function $f: X \mapsto Y$ is a triplet (X, Y, f) where X, Y are sets and f is a rule assigning to every element $x \in X$ exactly one element in Y, this element is denoted $f(x)$.
  - The key words here are "to every" and "exactly one". These are the two things we need to check when checking whether some f is a function or not.
- Composition.
  - Let $f: X \mapsto Y$, $g: Y \mapsto Z$ be functions. Then we can form the composite function $g \circ f: X \mapsto Z$ with $(g \circ f)(x)$ defined as $g(f(x))$. Note that here $x \in X$ so $f(x)$ is well-defined, and $f(x) \in Y$ so $g(f(x))$ is well-defined (As g is a function with domain Y, it must assign a value for $f(x) \in Y$).
  - It is important to keep in mind that in general

    $$g \circ f \neq f \circ g. \tag{1}$$

    In fact it may happen that only one of $g \circ f$ and $f \circ g$ could be defined at all.
- Inverse.
  - Let $f: X \mapsto Y$ be a function. We say $g: Y \mapsto X$ is its inverse function if and only if both of the following hold:
    - For every $x \in X$, $(g \circ f)(x) = x$;
    - For every $y \in Y$, $(f \circ g)(y) = y$.

    Note that $g \circ f: X \mapsto X$ and $f \circ g: Y \mapsto Y$.

    Exercise 1. Prove that if g is the inverse of f, then f is the inverse of g.

    Exercise 2. Prove that inverse is unique. That is if g_1, g_2 are both inverses of f, then $g_1 = g_2$ (for every $y \in Y$, $g_1(y) = g_2(y)$).
  - For a generic function, there are two obstacles for the existence of an inverse function g.
    1. There may be $x_1 \neq x_2$ with $f(x_1) = f(x_2)$. In this case if we let $y = f(x_1) = f(x_2)$, it is not possible to define $g(y)$, as it has to equal x_1 and also x_2.
    2. There may be $y \in Y - f(X)$. Again $g(y)$ cannot be defined. Because whatever element of X we assign $g(y)$ to, we would have $g(y) \in X$ and $f(g(y)) \in f(X)$ which means there always holds $y \neq f(g(y))$.
  - Inspired by this, we define: A function $f: X \mapsto Y$
    - is called "one-to-one" (or "injective") if and only if whenever $x_1 \neq x_2$, there holds $f(x_1) \neq f(x_2)$;

      Exercise 3. Understand that this is equivalent to: Whenever $f(x_1) = f(x_2)$ there holds $x_1 = x_2$.
    - is called "onto" (or "surjective") if and only if $f(X) = Y$.

      Exercise 4. Show that to prove $f(X) = Y$, it suffices to prove $Y \subseteq f(X)$, that is "for every $y \in Y$, there is $x \in X$ such that $y = f(x)$". (Hint:[1])

  – is called "bijective" if and only if it is one-to-one and onto.

Clearly when f is a bijection, we can define its inverse g.

○

THEOREM 1. *Let $f: X \mapsto Y$ and $g: Y \mapsto X$ be functions. Further assume g is the inverse of f. Then f is bijective.*

Proof. We prove that f is one-to-one and onto.

  – One-to-one.
    Let $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$. Since g is the inverse of f, we have

$$x_1 = (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2) = x_2. \tag{2}$$

  Therefore f is one-to-one.

  – Onto.
    Let $y \in Y$ be arbitrary. Set $x = g(y)$. Then we have

$$f(x) = f(g(y)) = (f \circ g)(y) = y. \tag{3}$$

  Therefore f is onto. □

- Important functions.

  ○ Polynomials.

$$P(x) = a_n x^n + \cdots + a_1 x + a_0 \tag{4}$$

  with $a_n \neq 0$. n is called the "degree" of the polynomial.

  ○ Rational functions.
    A function is called "rational" if it is the ratio between two polynomials.

  ○ Absolute value.

$$|x| := \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}. \tag{5}$$

  – Let $\max\{x, y\}$ denote the larger number between x, y, then

$$|x| = \max\{x, -x\}. \tag{6}$$

Exercise 5. Let $x \in \mathbb{R}$. If $x < 1$ and $-x < 1$ both holds, then $|x| < 1$.

Exercise 6. Let $x, y \in \mathbb{R}$. Prove that

$$\max\{x, y\} = \frac{x + y}{2} + \frac{|x - y|}{2}. \tag{7}$$

Find a similar formula for $\min\{x, y\}$ and justify.

  – Triangle inequality.
    Let $x, y \in \mathbb{R}$. Then

$$|x + y| \leqslant |x| + |y|. \tag{8}$$

Exercise 7. Let $x_1, \ldots, x_n \in \mathbb{R}$ where $n \in \mathbb{N}$ is arbitrary. Prove

$$|x_1 + \cdots + x_n| \leqslant |x_1| + \cdots + |x_n|. \tag{9}$$

1. $f(X) \subseteq Y$ is automatic.

Exercise 8. You are adding up 10 irrational numbers between 0 and 1 using your calculator which is accurate to the 8th digit. What is your estimate of the round-off error for the sum?