Math 117 Fall 2014 Lecture 10 (Sept. 18, 2014)

Reading: 314 Notes: Sets and Functions §3.1, §3.2.

- Function.
  - A function $f: X \mapsto Y$ is a triplet (X, Y, f) where X, Y are sets and f is a rule assigning to each element $x \in X$ exactly one element in Y, this element is denoted $f(x)$.
  - X: domain; Y: co-domain.
  - The only restriction is that f cannot assign to one x more than one $f(x)$.[1]
- Function and set relations.
  - Image and pre-image. Let $f: X \mapsto Y$ be a function and $A \subseteq X$, $S \subseteq Y$. Then the image of A under f is defined as $\{f(a) | a \in A\}$ and denoted $f(A)$. The pre-image of S under f is defined as $\{x \in X | f(x) \in S\}$ and denoted $f^{-1}(S)$.

    Notation. *Note that we will not use f^{-1} to denote inverse function.*
  - Proposition 1. *Let $f: X \mapsto Y$ be a function. Let $A, B \subseteq X, S, T \subseteq Y$. Further assume $A \subseteq B, S \subseteq T$. Then*

    a) $f(A) \subseteq f(B)$;

    b) $f^{-1}(S) \subseteq f^{-1}(T)$.

    Furthermore, $A \subset B$ does not imply $f(A) \subset f(B)$; and $S \subset T$ does not imply $f^{-1}(S) \subset f^{-1}(T)$.

    Proof. We prove a) and leave b) as exercise. Take an arbitrary $y \in f(A)$. By definition of $f(A)$ we have the existence of $a \in A$ such that $y = f(a)$. Now thanks to the fact that $A \subseteq B$ we have $a \in B$. By definition of $f(B)$ we know $f(a) \in f(B)$. But $y = f(a)$ so this gives $y \in f(B)$. Therefore $f(A) \subseteq f(B)$.

    To see that $A \subset B$ does not imply $f(A) \subset f(B)$, consider the following example: Let $f(x) = \sin x$, $A = [-\pi, \pi]$, $B = \mathbb{R}$. Then we have $A \subset B$ but $f(A) = f(B)$. □
- Function and set operations.

  Theorem 2. *Let $f: X \mapsto Y$ be a function. Let $A, B \subseteq X, S, T \subseteq Y$. Then*

  a) $f(A \cap B) \subseteq f(A) \cap f(B)$;

  b) $f(A \cup B) = f(A) \cup f(B)$;

  c) $f(A - B) \supseteq f(A) - f(B)$;

  d) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$;

  e) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$;

  f) $f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)$.

  - For proofs, see the note "Sets and Functions" for Math 314.

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1. If we further remove this restriction, we would have a "relation". Argue that relations from $\mathbb{R}$ to $\mathbb{R}$ can be identified with subsets of the plane $\mathbb{R}^2$.

- Note that here the naïve expection is that equality should hold for all six situations. Therefore it is important to construct examples where $f(A\cap B)\subset f(A)\cap f(B)$ and $f(A-B)\supset f(A)-f(B)$.

  Exercise 1. Find such examples. (Hint:[2])

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2. Consider $\sin x$.