Math 117 Fall 2014 Lecture 9 (Sept. 17, 2014)

Reading: 314 Notes: Sets and Functions §2.1 (Open and Closed Sets: Optional); Bowman §1.G.

- Operations on two sets (cont.)

  Example 1. Prove that $(A-B)\cap(B-A)=\varnothing$.

  Proof. Take an arbitrary $x\in(A-B)$. By definition $x\in A$, $x\notin B$. By definition of $B-A$ we have $x\notin B$ implies $x\notin B-A$. Therefore there is no x in both $A-B$ and $B-A$. By definition this gives $(A-B)\cap(B-A)=\varnothing$. □

  Example 2. Let $A\subseteq B$. Prove $A\cap C\subseteq B\cap C$ for any set C.

  Proof. Take an arbitrary $x\in A\cap C$. By defintion $x\in A$ and $x\in C$. Now as $A\subseteq B$, by definition of $\subseteq$ we conclude $x\in B$ from $x\in A$. Therefore $x\in B$ and $x\in C$ which gives $x\in B\cap C$. Thus we have proved $A\cap C\subseteq B\cap C$. □

  Example 3. Prove $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$.

  Proof. It suffices to prove both $A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)$ and $(A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)$.

  - $A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)$.
    Take an arbitrary $x\in A\cup(B\cap C)$. There are two cases.

    1. $x\in A$. By definition of $\cup$ there hold $x\in A\cup B$ and $x\in A\cup C$. Now by definition of $\cap$ we have $x\in(A\cup B)\cap(A\cup C)$.
    2. $x\notin A$. Then by definition of $\cup$ we must have $x\in B\cap C$. This means $x\in B$ and $x\in C$.
       Since $x\in B$, by definition of $\cup$ there holds $x\in A\cup B$. Similarly $x\in C$ implies $x\in A\cup C$.
       Finally by definition of $\cap$ we have $x\in(A\cup B)\cap(A\cup C)$.

    Thus we have proved if $x\in A\cup(B\cap C)$ then $x\in(A\cup B)\cap(A\cup C)$, which means $A\cup(B\cap C)\subseteq(A\cup B)\cap(A\cup C)$.

  - $(A\cup B)\cap(A\cup C)\subseteq A\cup(B\cap C)$.
    Left as exercise. □

  Please work on the examples and exercises in the assigned readings.

- Intervals.

  - Closed interval:
  $$[a,b]:=\{x|\, a\leqslant x\leqslant b\}; \tag{1}$$
  - Open interval:
  $$(a,b):=\{x|\, a<x<b\}; \tag{2}$$
  - Half open/half closed interval:
  $$[a,b):=\{x|\, a\leqslant x<b\}; \qquad (a,b]:=\{x|\, a<x\leqslant b\}. \tag{3}$$
  - Intervals involving infinity:
  $$(a,+\infty):=\{x|\, a<x\}; \qquad (-\infty,a):=\{x|\, x<a\}. \tag{4}$$

$[a, +\infty)$ and $(-\infty, a]$ can also be defined.

- Operations on more than two sets.

  DEFINITION 4. *Let W be a collection of sets. Then the union of all sets in this collection is defined as the set of those elements belonging to at least one set in W, and the intersection of all sets in this collection is defined as the set of those elements belonging to all the sets in W. That is*

$$\cup_{A\in W} A := \{x \mid \text{There is } A\in W \text{ such that } x\in A\}; \tag{5}$$

$$\cap_{A\in W} A := \{x \mid x\in A \text{ for every } A\in W\}. \tag{6}$$

  NOTATION. *Some times the sets in W can be "indexed", in this case we write the union/intersection slightly differently. For example, the intersection of all sets of the form $(1-x, 1)$ where x is some positive real number, can be written as*

$$\cap_{x>0}(1-x, 1). \tag{7}$$

  Example 5. Calculate

$$A := \cap_{n\in\mathbb{N}}\left[1-\frac{1}{n}, 1\right]; \qquad B := \cap_{n\in\mathbb{N}}\left(1-\frac{1}{n}, 1\right). \tag{8}$$

  Justify your result.
  Solution. First we guess the answers:

$$A = \{1\}, \qquad B = \varnothing. \tag{9}$$

  Now we justify them.

  - $A = \{1\}$.
    - First show $\{1\} \subseteq A$.

      Since $\{1\}$ has only one element, all we need to show is $1 \in A$. By definition of A if suffices to show $1 \in \left[1-\frac{1}{n}, 1\right]$ for every $n \in \mathbb{N}$.

      Let $n \in \mathbb{N}$ be arbitrary. Then we have

$$1-\frac{1}{n} \leqslant 1 \leqslant 1 \tag{10}$$

      which means $1 \in \left[1-\frac{1}{n}, 1\right]$.

    - Now we show $A \subseteq \{1\}$.

      Take an arbitrary $x \in A$. There are three cases.

      - $x = 1$. Then $x \in \{1\}$.
      - $x > 1$. In this case we have $x \notin [0, 1] = \left[1-\frac{1}{1}, 1\right]$. Therefore $x \notin A$. Contradiction. Thus this case is not possible.
      - $x < 1$. In this case we have $1 - x > 0$ and there is $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{1-x}$. This leads to $x < 1 - \frac{1}{n_0}$ which in turn gives

$$x \notin \left[1-\frac{1}{n_0}, 1\right] \tag{11}$$

        and consequently $x \notin A$. Thus this case is not possible either.

Summarizing, we see that every $x \in A$ also belongs to $\{1\}$, that is $A \subseteq \{1\}$.

– $B = \varnothing$.

The proof is almost identical to the $A \subseteq \{1\}$ part of the proof for $A = \{1\}$ and we leave it as an exercise.

Exercise 1. Calculate

$$C := \cup_{n \in \mathbb{N}} \left[1 - \frac{1}{n}, 1\right]; \qquad D := \cup_{n \in \mathbb{N}} \left(1 - \frac{1}{n}, 1\right). \tag{12}$$

Justify your results.

Exercise 2. Calculate

$$E := \cap_{n \in \mathbb{N}} \left(1 - \frac{1}{n}, 1 + \frac{1}{n^3}\right). \tag{13}$$

Justify your result.