Math 117 Fall 2014 Homework 2

Due Thursday Sept. 18 3pm in Assignment Box

Question 1. (5 pts)

a) (2 pts) *Find two irrational numbers a, b such that both $a+b$ and $a\times b$ are rational.*

b) (3 pts) *Can you find two irrational numbers a, b such that both $a+b$ and $a-b$ are rational? If you answer yes, find two such numbers; If you answer no, prove that this is not possible.*

Solution.

a) Take $a=1-\sqrt{2}, b=1+\sqrt{2}$ both irrational. Then we have $a+b=2, a\times b=-1$ both rational.

b) It is not possible. We prove by contradiction. Assume there are a, b irrational such that both $a+b$ and $a-b$ are rational. Then we can denote

$$a+b=\frac{p}{q}, \qquad a-b=\frac{r}{s} \tag{1}$$

with $p, q, r, s\in\mathbb{Z}$, $q, s>0$. Now we calculate

$$a=\frac{1}{2}\left[(a+b)+(a-b)\right]=\frac{p\,s+r\,q}{2\,q\,s}. \tag{2}$$

It is clear that $p\,s+r\,q, 2\,q\,s\in\mathbb{Z}$ and furthermore $2\,q\,s>0$. Dividing both numerator and denominator by their greatest common divisor we see that they can be made co-prime. Therefore $a\in\mathbb{Q}$, contradiction.

Note. It is OK to simply say if $a+b, a-b\in\mathbb{Q}$ then $a=\frac{1}{2}\left[(a+b)+(a-b)\right]\in\mathbb{Q}$.

Question 2. (5 pts) *Prove that $\sqrt{5}+\sqrt{11}$ is irrational.*

Proof. Assume the contrary, that is assume $\sqrt{5}+\sqrt{11}\in\mathbb{Q}$. Then so is

$$\sqrt{55}=\frac{1}{2}\left[\left(\sqrt{5}+\sqrt{11}\right)^2-16\right]. \tag{3}$$

Let $p, q\in\mathbb{Z}$, $q>0$, $(p, q)=1$ be such that $\sqrt{55}=\frac{p}{q}$. Then we have

$$p^2=55\,q^2 \tag{4}$$

which implies $5|\,p^2$. By the corrollary of the Fundamental Theorem of Arithmetic, we have $p=5\,k$ for some integer k. Substituting back into (4) we have

$$5\,p^2=11\,q^2. \tag{5}$$

This implies $5|\,11\,q^2$. As $5\not|\,11$, we must have $5|\,q^2$ which gives $5|\,q$. Thus 5 is a common divisor of both p, q, contradicting $(p, q)=1$. □

Remark. Another (very clever) proof by Peter: Assume $\sqrt{5}+\sqrt{11}$ is rational. Then so is

$$\sqrt{5}-\sqrt{11}=\frac{-6}{\sqrt{5}+\sqrt{11}}. \tag{6}$$

But this contradicts Q1 b) as both $\sqrt{5}$ and $\sqrt{11}$ are irrational.

QUESTION 3. (5 PTS) *Calculate (using a computing device if necessary)*

$$E_n := 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} \tag{7}$$

for $n = 3, 5, 7$. *For each* $n = 3, 5, 7$, *find the* ***smallest*** $m \in \mathbb{N}$ *such that*

$$\left(1 + \frac{1}{m}\right)^m \geqslant E_n. \tag{8}$$

Solution. We have

$$E_3 = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3} = 2.666\cdots \tag{9}$$

$$E_5 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} = 2.71666\cdots \tag{10}$$

$$E_7 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = \frac{685}{252} = 2.71825\underline{396825} \tag{11}$$

Note that 396825 is repeating.

To find the corresponding m we first try $m = 2^k$ until the value exceeds E_n and then refine the search by binary search algorithm[1]. Note that as $\left(1 + \frac{1}{m}\right)^m$ is increasing, all we need to find is m such that

$$\left(1 + \frac{1}{m-1}\right)^{m-1} < E_n \leqslant \left(1 + \frac{1}{m}\right)^m. \tag{12}$$

We do this through the help of http://keisan.casio.com/calculator.

- $n = 3$. $m = 26$.
- $n = 5$. $m = 841$.
- $n = 7$. $m = 48784$.

QUESTION 4. (5 PTS)

a) (3 PTS) *Prove that* $\left(1 + \frac{1}{n}\right)^{n+1}$ *is decreasing. (Hint:*[2] *)*

b) (2 PTS) *Use the result in a) to prove that* $\left(1 + \frac{1}{n}\right)^n$ *has a upper bound. Note that even if you cannot prove a), you still can apply the result in a) to b).*

Proof.

a) We prove

$$\left[\left(1 + \frac{1}{n}\right)^{n+1}\right]^{-1} = \left(\frac{n}{n+1}\right)^{n+1} = \left(1 - \frac{1}{n+1}\right)^{n+1} \tag{13}$$

1. See http://en.wikipedia.org/wiki/Binary_search_algorithm. On a computer, it will be faster to use Golden Section or (roughly equivalently) Fibonacci search http://en.wikipedia.org/wiki/Golden_section_search which will converge to the desired m in less steps. But binary search is obviously advantageous if no programming is intended.

2. Prove that $\left[\left(1 + \frac{1}{n}\right)^{n+1}\right]^{-1}$ is increasing.

is increasing. Calculate

$$
\begin{aligned}
\frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n}\right)^{n}} &= \left(1-\frac{1}{n+1}\right)\left[\frac{1-\frac{1}{n+1}}{1-\frac{1}{n}}\right]^{n} && (14)\\
&= \left(1-\frac{1}{n+1}\right)\left(\frac{n^2}{n^2-1}\right)^{n} && (15)\\
&= \left(1-\frac{1}{n+1}\right)\left(1+\frac{1}{n^2-1}\right)^{n} && (16)\\
&\geqslant \left(1-\frac{1}{n+1}\right)\left(1+\frac{n}{n^2-1}\right) && (17)\\
&= 1-\frac{1}{n+1}+\frac{n}{n^2-1}-\frac{n}{n+1}\frac{1}{n^2-1} && (18)\\
&= 1+\frac{1}{n^2-1}-\frac{n}{n+1}\frac{1}{n^2-1} && (19)\\
&= 1+\frac{1}{(n+1)(n^2-1)}>1. && (20)
\end{aligned}
$$

Note that (17) follows from (16) thanks to Bernoulli's inequality.

b) Thanks to a) we have

$$\left(1+\frac{1}{n}\right)^{n}<\left(1+\frac{1}{n}\right)^{n+1}<\left(1+\frac{1}{1}\right)^{1+1}=4 \tag{21}$$

for every $n \in \mathbb{N}$. Therefore 4 is a upper bound.

□