Automatic Linearity of Order Isomorphisms

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Natural Question

Suppose (X, C) and (Y, K) are partially ordered vector spaces and

$$f : X \to Y$$

is an *order isomorphism*, meaning that f is bijective and satisfies

$$x \geq y \iff f(x) \geq f(y), \ \forall x, y \in X.$$

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- On $\mathbb{R}$ the map $f(x) = x^3$ is an order isomorphism.
- On $C([0,1])$ with pointwise order the map

$$f(g(x)) = g(x)^3$$

for all $g \in C([0,1])$ and $x \in [0,1]$.

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- More recently, in 2011, S. Arstein-Avidan, B.A. Slomka fully characterized order isomorphisms on finite dimensional spaces.

Extreme Rays

Definition

A vector $r \in C$ is called an *extreme vector* (or *atom*) if for all $v \in C$ with $v \leq r$ there exists a $\lambda \in \mathbb{R}$ such that $v = \lambda r$.

- We denote the set of all extreme vectors of C by $\text{ext}(C)$.

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- An extreme ray in the direction of $r \in \mathrm{ext}(C)$ with apex $x \in X$ is defined as $x + \mathbb{R}^+ r$.
- Remark that for $r \in \mathrm{ext}(C)$ the interval $[0, r]$ is totally ordered.

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Theorem

Suppose (X, C) is Archimedean and let $x \in X$. A subset $H \subseteq X$ is an extreme ray with apex x if and only if it is maximal among subsets $G \subseteq \{x\}^u$ that satisfy:

(i) *G is directed, i.e., for all $y, z \in G$ there exists an $w \in G$ such that $w \geq y, z$.*

(ii) *For any $y, z \in G$ the order interval $[y, z]$ is totally ordered.*

(iii) *G contains two distinct points.*

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(iii) *G contains two distinct points.*

We conclude that f maps an extreme ray with apex $x \in X$ onto an extreme ray in Y with apex $f(x)$.

Additivity Along Extreme Rays

- Our first step towards showing that f is additive is to show that for all $x \in X$ and $r, s \in \mathrm{ext}(C)$ we have

$$f(x+r+s) - f(x+s) = f(x+r) - f(x).$$

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- It suffices to show that $f(x), f(x + r), f(x + s)$ and $f(x + r + s)$ are the corners of a parallelogram.
- Corresponding to f there exists a bijection $\varphi : E_C \to E_K$, where E_C and E_K denote the collections of extreme rays of C and K, respectively, such that

$$f(x + R) = f(x) + \varphi(R), \qquad x \in X,\ R \in E_C.$$

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- A set $U \subseteq X$ is called an upper set if it satisfies $U = U^u$.
- Important examples of upper sets are X, C and C°.

Diagonal Form of Order Isomorphisms

Theorem

Suppose (X, C) is Archimedean. For any collection of extreme vectors $(v_\alpha)_{\alpha\in I}$ there exist increasing and bijective $f_\alpha : \mathbb{R} \to \mathbb{R}$ such that for any $x \in X$ and $v = \sum_{i=1}^{n} \lambda_i v_{\alpha_i}$ (with $(x+v) \in U$) we have

$$f(x+v) = f(x) + \sum_{i=1}^{n} f_{\alpha_i}(\lambda_i)\tau(x, v_{\alpha_i}),$$

where $\tau(x, v_{\alpha_i}) := f(x + v_{\alpha_i}) - f(x) \in \varphi(\mathbb{R}^+ v_{\alpha_i})$.

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In the case that $f : C \to K$, we get $f(v) = \sum_{i=1}^n f_{\alpha_i}(\lambda_i) f(v_{\alpha_i})$.

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- When does this diagonal form imply that the order isomorphism is affine linear?
- $\tau(x, v_\alpha)$ needs to be constant in its first argument and all f_α need to be equal to the identity on $\mathbb{R}$.

Engaged Extreme Rays

Definition

Suppose $\mathcal{R}$ is a collection of rays (with apex 0). A ray $R \in \mathcal{R}$ is called *engaged* if $R \subseteq \text{span}(\mathcal{R} \setminus \{R\})$ holds and is called *disengaged* otherwise.

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- Let C_E denote the cone spanned by the engaged extreme rays of C and $X_E := C_E - C_E$.
- All extreme rays of C_E are engaged.
- We apply the diagonal form of order isomorphisms to a collection of engaged extreme vectors.
- **Conclusion**: f is affine linear on the linear subspace spanned by the engaged extreme rays.

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- We can show that f is affine linear on the supremum closure of X_E.
- Moreover, f is affine linear on the supremum closure of the infimum closure of X_E.

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- Consider $X = B(H)_{sa}$ equipped with $A \geq B :\Leftrightarrow \langle Ax, x\rangle \geq \langle Bx, x\rangle$ for all $x \in H$.

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- Using linear order isomorphism $T_A(B) := A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ we can show that any invertible $A \in B(H)_{sa}^{+}$ is contained in the supremum closure of Y.
- Lastly, we use that $A + \lambda I$ is invertible and positive for some λ.