On Dunford-Pettis-like functions on Banach lattices

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Outline of talk

- Introduce some preliminary notation and definitions.
- Introduce and characterise the disjoint Dunford-Pettis (Dunford-Pettis*) property of order p.
- Introduce disjoint p-convergent functions.
- Provide conditions under which polynomials as well as symmetric separately compact bilinear maps are disjoint p-convergent.

Main Literature Sources I

AB C. D. ALIPRANTIS AND O. BURKINSHAW, Positive Operators, *Springer, Dordrecht*, (2006) (reprint of the 1985 original copy).

CGL H. CARRION, P. GALINDO AND M.L. LOURENCO, A stronger Dunford-Pettis property, *Studia Mathematica*, 2008, **3**:205–216.

CCJ2 J. X. CHEN, Z. L. CHEN AND G. X. JI, Almost limited sets in Banach lattices, *J.Math.Anal.Appl.* **412** (2014), 547–553.

D S. DINEEN, Complex analysis on infinite dimensional spaces, *Springer–Verlag*, 1981.

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Preliminaries I

- A mapping $P : X \to Y$ is called an n-homogeneous polynomial from X to Y if there exists an element $L \in \mathcal{L}_a(^nX; Y)$ such that $P = L \circ \Delta_n$; that is we have $P(x) = L(x, x, \ldots, x)$ for all $x \in X$.
- We let $\mathcal{P}(^nX; Y)$ denote the space of continuous n-homogeneous polynomials from X into Y.
- Let U be an open subset of X. A mapping $f : U \to Y$ is said to be holomorphic (or analytic) if for each $a \in U$ there exists a ball $B(a, r) \subset U$ and a sequence of polynomials $P_m \in \mathcal{P}(^nX; Y)$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - a),$$

uniformly on $B(a, r)$.

Preliminaries II

- The vector space of all holomorphic mappings from U into Y is denoted by $\mathcal{H}(U;Y)$. When $Y=\mathbb{C}$, we write $\mathcal{H}(U;\mathbb{C})=\mathcal{H}(U)$.
- $\mathcal{P}(X;Y)\subset\mathcal{H}(X;Y)$, i.e. each polynomial (as a linear combination of m-homogeneous polynomials) is holomorphic.
- For any unexplained terminology we refer the listener to the book of Dineen.

Definition

- A subset A of X is said to be a Dunford-Pettis (respectively limited) set if for all $(x_n^*) \in c_0^{weak}(X^*)$ (respectively $(x_n^*) \in c_0^{weak^*}(X^*)$), we have that $\sup_{x \in A} |x_n^*(x)| \xrightarrow[\infty]{n} 0$.
- A sequence (x_n) is said to be Dunford-Pettis (respectively, limited) if the set $\{x_n : n \in \mathbb{N}\}$ is Dunford-Pettis (respectively, limited).

Definition: Disjoint DPP_p (Motivated by Chen, et. al.)

A Banach lattice E is said to have the disjoint Dunford-Pettis property of order p (disjoint DPP_p for short) if every disjoint weakly p-summable sequence in E is Dunford-Pettis.

Theorem 1

A Banach lattice E has the disjoint DPP_p if and only if $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ and all $(x_n^*) \in c_0^{weak}(E^*)$.

Proof I

Suppose E has the disjoint DPP_p.

Let $(x_n) \in \ell_p^{weak}(E)$ be a disjoint sequence.

By the assumption the set $\{x_n : n \in \mathbb{N}\}$ is Dunford-Pettis.

Thus, for $(x_n^*) \in c_0^{weak}(E^*)$ we have $\sup_k |x_n^*(x_k)| \xrightarrow[\infty]{n} 0$.

In particular, $x_n^*(x_n) \xrightarrow[\infty]{n} 0$.

Conversely, assume $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ for all $(x_n^*) \in c_0^{weak}(E^*)$ and all disjoint sequences $(x_n) \in \ell_p^{weak}(E)$.

Suppose E does not have the disjoint DPP_p.

Proof II

Then there exists a disjoint $(x_n) \in \ell_p^{weak}(E)$ which is not Dunford-Pettis, i.e. there exists $(x_n^*) \in c_0^{weak}(E^*)$ such that

$$\sup_k |x_n^*(x_k)| \nrightarrow 0 \text{ if } n \to \infty.$$

Thus we may find a subsequence of (x_n^*), which we denote by (x_n^*) again, such that for all $n \in \mathbb{N}$,

$$\sup_k |x_n^*(x_k)| \geq \varepsilon. \tag{1}$$

Let $n_1 = 1$ and let $x_{k_1} \in \{x_n : n \in \mathbb{N}\}$ be such that

$$|x_n^*(x_{k_1})| \geq \frac{\varepsilon}{2}.$$

Proof III

Since $x_n^*(x_j) \xrightarrow[\infty]{n} 0$ for all $1 \leq j \leq k_1$, we may choose an index $n_2 > n_1$ such that $|x_{n_2}^*(x_j)| < \frac{\varepsilon}{2}$ for all $j = 1, 2, \cdots k_1$.

By (1) there has to be $x_{k_2} \in \{x_n : n \in \mathbb{N}\}$ such that $|x_{n_2}^*(x_{k_2})| \geq \frac{\varepsilon}{2}$ and where $k_2 > k_1$.

Similarly, since $x_n^*(x_j) \xrightarrow[\infty]{n} 0$ for all $1 \leq j \leq k_2$, we can find an index $n_3 > n_2 > n_1$ so that

$$|x_{n_3}^*(x_j)| < \frac{\varepsilon}{2} \text{ for } j = 1, 2, \cdots k_2.$$

Again by (1) there has to be $x_{k_3} \in \{x_n : n \in \mathbb{N}\}$ such that $|x_{n_3}^*(x_{k_3})| \geq \frac{\varepsilon}{2}$ and where $k_3 > k_2 > k_1$.

Proof IV

Continuing in this way, we may construct two "new" sequences $(x_n^*) \in c_0^{weak}(E^*)$ and $(x_n) \in \ell_p^{weak}(E)$ (where the x_n's are mutually disjoint) such that

$$|x_n^*(x_n)| \geq \frac{\varepsilon}{2}.$$

This contradicts our assumption.

Definition: Disjoint DP^*P_p

A Banach lattice E is said to have the disjoint Dunford-Pettis* property of order p (disjoint DP^*P_p for short) if every disjoint weakly p-summable sequence in E is limited.

Theorem 2

A Banach lattice E has the disjoint DP^*P_p if and only if $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ and all $(x_n^*) \in c_0^{weak^*}(E^*)$.

Definition: Disjoint p-convergent operator

An operator $T : E \to X$ from a Banach lattice to a Banach space is said to be disjoint p-convergent if for every disjoint sequence $(x_n) \in \ell_p^{weak}(E)$, we have that $\|Tx_n\| \xrightarrow[\infty]{n} 0$.

Proposition 3

Let $1 \leq p < \infty$. A Banach lattice E has the disjoint DP^*P_p if and only if every operator $T \in \mathcal{L}(E, c_0)$ is disjoint p-convergent.

Definitions

- A function f from a Banach space X into a Banach space Y is said to be completely continuous if it maps weakly convergent sequences into convergent sequences. (Carrion, et. al.)
- Let $1 \leq p < \infty$. A function f from a Banach space X into a Banach space Y is said to be p-convergent if it maps weak p-convergent sequences onto norm convergent sequences. (Fourie and Zeekoei)

Definition: Disjoint p-convergent function

Let $1 \leq p < \infty$. A function f from a Banach lattice E into a Banach space X is said to be disjoint p-convergent if it maps disjoint weak p-convergent sequences onto norm convergent sequences.

Proposition 4

Let $1 \leq p < \infty$, E be a Banach lattice and X be a Banach space and assume that X contains an isomorphic copy of c_0. If every $T \in \mathcal{L}(E, X)$ is disjoint p-convergent, then E has the disjoint DP^*P_p. In this case, every polynomial $P \in \mathcal{P}(^nE, X)$ is a disjoint p-convergent function for all $n \in \mathbb{N}$.

Definition

We call a subset A of a Banach lattice E "disjoint weakly p-compact" if it is weakly p-compact and its elements are mutually disjoint.

Proposition 5

A Banach lattice E has the disjoint DP^*P_p if and only if all disjoint weakly p-compact sets in E are limited.

Corollary 6

Let E be a Banach lattice with the disjoint DP^*P_p and let $(P_n) \subset \mathcal{P}(^kE)$ such that $P_n \xrightarrow[\infty]{n} P \in \mathcal{P}(^kE)$ pointwise. Then $P_n \xrightarrow[\infty]{n} P$ uniformly on all disjoint weakly p-compact sets in E and for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$, we have $P_n(x_n) \xrightarrow[\infty]{n} 0$.

Proposition 7

Let E be a Banach lattice. Then following assertions are equivalent:

(1) E has the disjoint DP^*P_p.

(2) Every operator $T : E \to c_0$ is disjoint p-convergent.

(3) For all integers k, each polynomial $P \in \mathcal{P}(^kE, c_0)$ is disjoint p-convergent.

(4) For some integer k, each polynomial $P \in \mathcal{P}(^kE, c_0)$ is disjoint p-convergent.

Proposition 8

If E has the disjoint DP^*P_p and X is a Gelfand-Phillips space, then every $P \in \mathcal{P}(^nE, X)$ is disjoint p-convergent. Furthermore, each $f \in \mathcal{H}(E, X)$ which is bounded on limited sets, is weakly continuous on disjoint weakly p-compact sets.

Remark

Let E, F be Banach lattices. Recall that the ordering on $E \times F$ is defined by

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

In this ordering we have:

(1) $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$;

(2) $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$;

(3) $|(x_1, y_1)| = (|x_1|, |x_2|)$.

Lemma 9

Let E, F be Banach lattices and $(x_n) \in E^{\mathbb{N}}$, $(y_n) \in F^{\mathbb{N}}$ Then:

(i) (x_n, y_n) is a disjoint sequence in $E \times F$ if and only if (x_n) is a disjoint sequence in E and (y_n) is a disjoint sequence in F.

(ii) $(x_n, y_n) \in \ell_p^{weak}(E \times F)$ if and only if $(x_n) \in \ell_p^{weak}(E)$ and $(y_n) \in \ell_p^{weak}(F)$.

Definition

Let X, Y and Z be Banach spaces. A bilinear operator

$$\phi : X \times Y \to Z$$

is called separately compact if for each fixed $y \in Y$, the linear operator $T_y : X \to Z : \ x \mapsto \phi(x, y)$ is compact and for each fixed $x \in X$, the linear operator $T_x : Y \to Z : \ y \mapsto \phi(x, y)$ is compact.

Lemma 10

If $T \in \mathcal{L}(X, c_0)$ and $T \otimes T : X \times X \to c_0$ is given by

$$(T \otimes T)(x, y) = T(x)T(y),$$

the coordinate-wise product of two sequences in c_0, then $T \otimes T$ is a separately compact bilinear operator.

Proposition 11

Let E be a Banach lattice. If every symmetric bilinear separately compact map $E \times E \to c_0$ is disjoint p-convergent, then E has the disjoint DP^*P_p.

Proof I

Consider any $T \in \mathcal{L}(E, c_0)$.

Let $T \otimes T : X \times X \to c_0$ be the separately compact bilinear operator as in Lemma 10 above.

By our assumption, $T \otimes T$ is therefore also disjoint p-convergent.

Consider any disjoint sequence $(x_i) \in \ell_p^{weak}(E)$.

We prove that

$$\|Tx_i\| \longrightarrow 0, \text{ if } i \longrightarrow \infty :$$

By Lemma 9, $(x_n, x_n)_n \in \ell_p^{weak}(E \times E)$ and (x_n, x_n) is a disjoint sequence.

Proof II

Since $T \otimes T$ is disjoint p-convergent, it follows that

$$\begin{aligned} \|Tx_n\|_{c_0}^2 &= \|(T \otimes T)(x_n, x_n)\|_{c_0} \\ &\xrightarrow[\infty]{n} \|(T \otimes T)(0,0)\|_{c_0} \\ &= 0. \end{aligned}$$

By Proposition 3, E has the disjoint DP^*P_p.

Thank you for your attendance and attention.