Positive linear maps on C*-algebras-Choi's conjecture-

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Definition

(i) A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be *selfadjoint* if $\Phi(X^*) = \Phi(X)^*$ for every $X \in \mathcal{A}$, *positive* if $\Phi(A) \geq 0$ for every $A \geq 0$, and *unital* if $\Phi(I) = I$.

(ii) A linear map Φ is called a *C^*-homomorphism* if $\Phi(X^2) = (\Phi(X))^2$ for all $X \in \mathcal{A}$.

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$\Leftrightarrow \Phi(A^n) = (\Phi(A))^n$ for all $A \in \mathcal{A}_s$ and for all $n = 1, 2, 3, \cdots$.
$\Leftrightarrow$ for each real continuous function $h(t)$ defined on an interval of the real axis,

$$\Phi(h(A)) = h(\Phi(A)) \text{ for all } A \in \mathcal{A}_s,$$

provided the both sides are well-defined.

Remark

A C^*-homomorphism Φ is not necessarily an algebraic homomorphism (for instance, transposed mapping), but Φ is positive and continuous, and $\Phi(I)$ is a projection. (see Størmer[1963] for its characterization)

Definition

A real continuous function f defined on an interval J is called an operator convex function if

$$f(sA + (1-s)B) \leq sf(A) + (1-s)f(B)$$

for all $A, B \in \mathcal{A}_s$ with $\sigma(A),\ \sigma(B) \subset J$ and for all $s: 0 < s < 1$.

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(i) t and t^2 are operator convex on $(-\infty, \infty)$.

(ii) $\frac{1}{t}$ is operator convex on $(0, \infty)$.

(iii) t^λ $(1 < \lambda < 2)$ is operator convex on $(0, \infty)$.
$(\because)$ $t^{1+a} = \frac{\sin a\pi}{\pi} \int_0^\infty (t - x + \frac{x^2}{x+t}) x^{a-1} dx$ $(0 < a < 1)$

(iv) t^λ $(-1 < \lambda < 0)$ is operator convex on $(0, \infty)$.
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(Davis [1952], Choi [1974]) If f is an operator convex function on $(-a, a)$ and if a positive linear map Φ is unital, then

$$f(\Phi(A)) \leq \Phi(f(A))$$

for all $A \in \mathcal{A}_s$ s.t. $\sigma(A) \subset (-a, a)$.

(Kadison [1952]) If a linear map Φ is unital and if

$$\Phi(|A|) = |\Phi(A)|$$

for all $A \in \mathcal{A}_s$, then Φ is a C^*-homomorphism.

(Choi [1974]) Let f be a non-affine operator convex function on $(-a, a)$ and Φ a unital and positive linear map.
Then Φ is a C^*-homomorphism if

$$f(\Phi(A)) = \Phi(f(A)) \tag{1}$$

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$*$ We shall prove his conjecture.

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Theorem 1

Let $\Phi \in B(\mathcal{A}, \mathcal{B})$ be a selfadjoint -not necessarily positive- and unital map and f a non-affine continuous function on an interval J. If for a given $A \in \mathcal{A}_s$

$$\Phi(f(X)) = f(\Phi(X)) \tag{2}$$

for all $X \in \mathcal{L}(A)$ s.t. $\sigma(X), \sigma(\Phi(X)) \subset J$, then $\Phi(A^2) = \Phi(A)^2$.

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Remark. Suppose s is an interior point of J. Then for $A \in \mathcal{A}_s$ the spectra of $X = tA + sI$ and $\Phi(X) = t\Phi(A) + sI$ are both in J for sufficiently small t.

Corollary 2

Let Φ be a linear, positive and unital map, and let f a non-affine continuous function on J.

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(i) *If for a given $A \in \mathcal{A}_s$*
$\Phi(f(X)) = f(\Phi(X))$ for all $X \in \mathcal{L}(A)$ s.t. $\sigma(X) \subset J$,
then $\Phi(A^n) = \Phi(A)^n$ for $n \in N$.

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(i) *If for a given $A \in \mathcal{A}_s$*
$\Phi(f(X)) = f(\Phi(X))$ *for all* $X \in \mathcal{L}(A)$ *s.t.* $\sigma(X) \subset J$,
then $\Phi(A^n) = \Phi(A)^n$ *for* $n \in N$.

(ii) *If* $\Phi(f(X)) = f(\Phi(X))$ *for all* $X \in \mathcal{A}_s$ *s.t.* $\sigma(X), \sigma(\Phi(X)) \subset J$, *then* Φ *is a* C^**-homomorphism on* $\mathcal{A}$.*(Choi's conjecture)*

Proof of (i) of Corollary 2.

By Theorem, we get $\Phi(A^2) = \Phi(A)^2$. Suppose B commutes to A. From Kadison's inequality

$$\Phi\left((A + tB)^2\right) \geq (\Phi(A) + t\Phi(B))^2 \quad (\forall t \in \mathbb{R}).$$

This implies $\phi(A)\phi(B) + \phi(B)\phi(A) = 2\phi(AB)$ because of $\Phi(A^2) = \Phi(A)^2$.

This gives $\phi(A^n) = \phi(A)^n$ for every n. $\square$

Proof of Theorem 1

Lemma 3

If $f(t)$ is twice differentiable at an interior point $a \in J$ and $f''(a) \neq 0$, then Theorem 1 holds for f.

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Proof of Lemma 1

From the formula

$$f''(a) = \lim_{t \to 0} \frac{f(a+t) + f(a-t) - 2f(a)}{t^2},$$

$n^2 \left(f(a + \frac{1}{n}t) + f(a - \frac{1}{n}t) - 2f(a) \right) \Longrightarrow f''(a)t^2$
on every finite interval as $n \to \infty$.

Suppose $\Phi(f(X)) = f(\Phi(X))$ on $\mathcal{L}(A)$. Since $\sigma(aI \pm \frac{1}{n}A), \sigma(aI \pm \frac{1}{n}\Phi(A)) \subset J$ for sufficiently large n,

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$$\begin{aligned}&\Phi\left(n^2\left(f(aI+\frac{1}{n}A)+f(aI-\frac{1}{n}A)-2f(aI)\right)\right)\\=&n^2\left(f(aI+\frac{1}{n}\Phi(A))+f(aI-\frac{1}{n}\Phi(A))-2f(aI)\right).\end{aligned}$$

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$$\Phi\left(n^2\left(f(aI + \frac{1}{n}A) + f(aI - \frac{1}{n}A) - 2f(aI)\right)\right)$$
$$= n^2\left(f(aI + \frac{1}{n}\Phi(A)) + f(aI - \frac{1}{n}\Phi(A)) - 2f(aI)\right).$$

By letting $n \to \infty$, we derive $\Phi(f''(a)A^2) = f''(a)\Phi(A)^2$ and hence $\Phi(A^2) = \Phi(A)^2$. $\square$

Lemma 4

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(ii) *If f is non-affine on J, then for every $\delta > 0$ there is a subinterval $[a, b]$ of J such that $b - a < \delta$ and the restriction of f to $[a, b]$ is non-affine.*

It is enough for us to prove that Theorem 1 holds for a non-affine continuous function f defined on a finite closed interval $[a, b]$.

Mollifier (Regularization)

Let $\varphi(t)$ be a C^∞ function on $\mathbb{R}$ with the support in $[-1, 1]$ such that $\varphi(t) \geq 0$ and $\int_{-\infty}^{\infty} \varphi(t)dt = 1$.

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For a continuous function $f(t)$ on $[a, b]$ and $\lambda > 0$, $f_\lambda(t)$ on $\mathbb{R}$ defined by

$$f_\lambda(t) = \frac{1}{\lambda} \int_a^b \varphi(\frac{t-s}{\lambda}) f(s) ds$$

is a C^∞ function and the support of f_λ is in $[a - \lambda, b + \lambda]$.

Since

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and

$$f_\lambda(X) = \int_{-1}^{1} \varphi(x) f(X - \lambda x I) dx \tag{3}$$

for $X \in \mathcal{A}_s$ with $\sigma(X) \subset (a + \lambda, b - \lambda)$, where the integral converges in the operator norm.

Lemma 5

Let $f(t)$ *be a function on* $[a, b]$*, and define* $f_\lambda(t)$ *as above for* $0 < \lambda < (b-a)/2$*.*
If there is a λ *such that Theorem 1 holds for* $f_\lambda|_{[a+\lambda, b-\lambda]}$*, then it does for* $f(t)$ *itself.*

Lemma 5

Let $f(t)$ be a function on $[a, b]$, and define $f_\lambda(t)$ as above for $0 < \lambda < (b-a)/2$.
If there is a λ such that Theorem 1 holds for $f_\lambda|_{[a+\lambda, b-\lambda]}$, then it does for $f(t)$ itself.

Proof Fix λ so that Theorem 1 holds for $f_\lambda|_{[a+\lambda, b-\lambda]}$.
Let Φ be a linear, continuous, selfadjoint and unital map, and $A \in \mathcal{A}_s$. Assume

$$\Phi(f(X)) = f(\Phi(X)) \tag{4}$$

for all $X \in \mathcal{L}(A)$: $\sigma(X), \sigma(\Phi(X)) \subseteq [a, b]$.
We have to show that $\Phi(A^2) = \Phi(A)^2$.

Take $\forall Y \in \mathcal{L}(A)$ with $\sigma(Y),\ \sigma(\Phi(Y)) \subseteq [a+\lambda,\ b-\lambda]$.
Then by (3),

$$f_\lambda(Y) = \int_{-1}^{1} \varphi(x) f(Y - \lambda x I) dx,$$
$$f_\lambda(\Phi(Y)) = \int_{-1}^{1} \varphi(x) f(\Phi(Y) - \lambda x I) dx.$$

Take $\forall Y \in \mathcal{L}(A)$ with $\sigma(Y),\ \sigma(\Phi(Y)) \subseteq [a+\lambda,\ b-\lambda]$.
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Since, for $-1 \leq x \leq 1$, $Y - \lambda x I \in \mathcal{L}(A)$ and
$\sigma(Y - \lambda x I),\ \sigma(\Phi(Y - \lambda x I)) \subseteq [a, b]$,
in view of (4),
$\Phi(f(Y - \lambda x I)) = f(\Phi(Y - \lambda x I)) = f(\Phi(Y) - \lambda x I)$.

This yields

$$\Phi(f_\lambda(Y)) = \int_{-1}^{1} \varphi(x)\Phi(f(Y - \lambda x I))dx$$

$$= \int_{-1}^{1} \varphi(x) f(\Phi(Y) - \lambda x I)dx = f_\lambda(\Phi(Y)).$$

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$$= \int_{-1}^{1} \varphi(x) f(\Phi(Y) - \lambda x I)dx = f_\lambda(\Phi(Y)).$$

We have therefore shown that

$$\Phi(f_\lambda(Y)) = f_\lambda(\Phi(Y))$$

for all $Y \in \mathcal{L}(A)$ with $\sigma(Y),\ \sigma(\Phi(Y)) \subseteq [a+\lambda,\ b-\lambda]$. By the assumption about $f_\lambda|_{[a+\lambda,\ b-\lambda]}$ we have $\Phi(A^2) = \Phi(A)^2$. $\square$

Lemma 6

Let $f(t)$ be a non-affine continuous function defined on $[a,\ b]$. Then there is a function $g \in C_0^\infty(a,b)$ such that $\int_a^b g''(t)f(t)dt \neq 0$.

Lemma 6

Let $f(t)$ be a non-affine continuous function defined on $[a,\ b]$. Then there is a function $g \in C_0^\infty(a,b)$ such that $\int_a^b g''(t)f(t)dt \neq 0$.

Proof Assume that $\int_a^b g''(t)f(t)dt = 0$ for every $g \in C_0^\infty(a,b)$. By considering f as a distribution on (a,b) we get

$$\int_a^b g(t)f''(t)dt = \int_a^b g''(t)f(t)dt = 0,$$

which means $f'' = 0$. Hence, $f(t) = c_1 t + c_2$ as distribution (L. Schwartz).

This implies

$$\int_a^b (f(t) - c_1 t - c_2) h(t) dt = 0 \quad \text{for all } h \in C_0^\infty(a, b).$$

Since $f(t)$ is continuous, $f(t) - c_1 t - c_2 = 0$ on (a, b) and hence on $[a, b]$; $f(t)$ is therefore affine; contradiction. $\square$

Proof of Theorem

Take a g given in Lemma 6 and suppose its support is contained in $(a+\delta, b-\delta)$ for $\delta > 0$. Then for f_λ

$$\int_{a+\delta}^{b-\delta} g(t) f_\lambda''(t)dt = \int_{a+\delta}^{b-\delta} g''(t) f_\lambda(t)dt$$

$$\overset{\lambda\downarrow 0}{\rightarrow} \int_{a+\delta}^{b-\delta} g''(t)f(t)dt = \int_a^b g''(t)f(t)dt \neq 0.$$

Hence, for sufficiently small $\lambda > 0$, $f_\lambda''(t) \not\equiv 0$ on $[a+\delta, b-\delta]$. Thus for sufficiently small λ less than δ, we have $f_\lambda''(t) \not\equiv 0$ on $[a+\lambda, b-\lambda]$. By Lemma 3, Theorem 1 holds for $f_\lambda|_{[a+\lambda, b-\lambda]}$; hence by Lemma 5 it does for f too. This concludes the proof of Theorem 1.
$\square$

Thank you !

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