The Regular Algebra Numerical Range

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Hilbert Spaces

Definition

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is defined as

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

Theorem (Toeplitz-Hausdorff)

The numerical range of every bounded linear operator T on a Hilbert space is convex.

Theorem

If T is a bounded linear operator on a Hilbert space $\mathcal{H}$, then the spectrum of T is contained in the closure of the numerical range of T.

Banach Spaces

Definition

For a linear operator $T \in \mathcal{L}(E)$ on a Banach space E, the algebraic numerical range of T is defined to be

$$V(\mathcal{L}(E), T) := \{\Phi(T) : \Phi \in \mathcal{L}(E)^*, \Phi(I) = 1 = \|\Phi\|\}$$

Such Φ are called states.

Definition

For a linear operator $T \in \mathcal{L}(E)$ on a Banach space E, the spatial numerical range of T is defined to be

$$V(T) := \{f(Tx) : x \in E, f \in E^*, \|x\| = \|f\| = 1 = f(x)\}$$

Basic Properties

- $V(T) \subseteq V(\mathcal{L}(E), T)$
- $\sup\{|\lambda| : \lambda \in V(T)\} = \sup\{|\lambda| : \lambda \in V(\mathcal{L}(E), T)\}$
  This supremum is known as the numerical radius, often denoted $v(T)$. In a complex Banach space, the numerical radius is equivalent to the norm:
  $$\frac{1}{e}\|T\| \leq v(T) \leq \|T\|$$

Theorem (Williams)

For $T \in \mathcal{L}(E)$ we have that

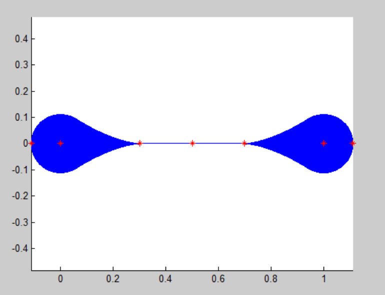
$$\sigma(T) \subseteq \overline{V(T)}$$

Basic Properties

- $V(\mathcal{L}(E), T)$ is closed and convex, but $V(T)$ need not be. In fact $\overline{co}V(T) = V(\mathcal{L}(E), T)$

Examples

Let $T = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}$ and $\|(z,w)\| = \max\{\|(z,w)\|_\infty, \frac{3}{\sqrt{10}}\|(z,w)\|_2\}$
Then the following picture represents $V(T)$.



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Regular Algebra Numerical Range

Let E be a Dedekind complete Banach Lattice over $\mathbb{C}$. Also, let $\mathcal{L}_r(E)$ be the set of all regular operators on E, the operators that are the difference of two positive linear operators.

Definition

For a regular operator $T \in \mathcal{L}_r(E)$ on a Banach Lattice E, the regular algebra numerical range is defined to be

$$V(\mathcal{L}_r(E), T) := \{\Phi(T) : \Phi \in \mathcal{L}_r(E)^*, \|\Phi\| = 1 = \Phi(I)\}$$

Such Φ will be called regular states.

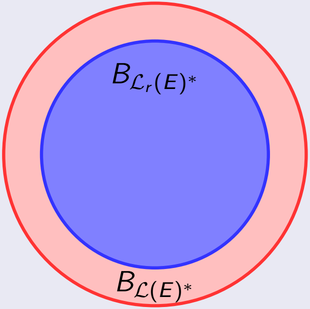
Relations to Classical Numerical Ranges

For $T \in \mathcal{L}_r(E)$ we have

$$V(T) \subseteq V(\mathcal{L}(E), T) \subseteq V(\mathcal{L}_r(E), T)$$

Proof.

Note that for $\Phi \in \mathcal{L}_r(E)^*$ we have that $\|\Phi\| \leq \|\Phi\|_r$. This implies the unit balls of each space look like:



□

T disjoint with the identity

Theorem

For $T \in \mathcal{L}_r(E)$ and $T \perp I$ we have that $V(\mathcal{L}_r(E), T)$ is a disk centered around $z = 0$.

We will see an example of an operator that is not disjoint with the identity, but still has a disk centered around $z = 0$ as its numerical range.

Property

Let $\mathcal{H}$ be a Hilbert space such that $H = H_1 \oplus_2 H_2$. Then

$$V(\mathcal{L}_r(\mathcal{H}), T_1 \oplus T_2) = co\{V(\mathcal{L}_r(H_1), T_1), V(\mathcal{L}_r(H_2), T_2)\}$$

T disjoint with the identity

Theorem

For $T \in \mathcal{L}_r(E)$ and $T \perp I$ we have that $V(\mathcal{L}_r(E), T)$ is a disk centered around $z = 0$.

Examples

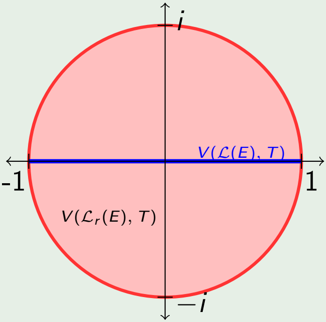
Let $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus [1]$ on $\ell_2(\mathbb{C})$.

$$\begin{aligned} V(\mathcal{L}_r(\mathcal{H}), T) &= co\left\{ V\left(\mathcal{L}_r(\mathcal{H}_1), \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right), V(\mathcal{L}_r(\mathcal{H}_2), [1]) \right\} \\ &= co\{\text{unit disk}, 1\} \\ &= \text{unit disk} \end{aligned}$$

An example of $V(\mathcal{L}(E), T) \neq V(\mathcal{L}_r(E), T)$

Examples

Let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over $\ell_2(\mathbb{C})$. Since E is a Hilbert space we can use the fact that $V(T) = W(T) = \{\langle Tx, x\rangle : x \in E, \|x\| = 1\} \subseteq \mathbb{R}$. Furthermore, $V(\mathcal{L}(E), T) = \overline{co}V(T) \subseteq \mathbb{R}$.
However, $T \perp I$, so $V(\mathcal{L}_r(E), T)$ must be a disk and thus not a subset of $\mathbb{R}$.



T in the center, $\mathcal{Z}(E)$

For the Dedekind complete Banach lattice, E, let $\mathcal{Z}(E) = \{T \in \mathcal{L}_r(E) : |T| \leq \lambda I\}$ be the center of E.

Theorem

For $T \in \mathcal{Z}(E)$ we have that $V(\mathcal{L}_r(E), T) = V(\mathcal{Z}(E), T) = co\,(\sigma(T))$.

Property

Let $\Phi \in \mathcal{L}_r(E)^*$ be a regular state. Then $\Phi\,|_{\mathcal{Z}(E)} \geq 0$.

Theorem

For $T \in \mathcal{L}_r(E)$ we have that $V(\mathcal{L}_r(E), T) \subseteq \mathbb{R}^+$ if and only if $T \geq 0$ and $T \in \mathcal{Z}(E)$.

T in the center, $\mathcal{Z}(E)$

Theorem

For $T \in \mathcal{L}_r(E)$ we have that $V(\mathcal{L}_r(E), T) \subseteq \mathbb{R}^+$ if and only if $T \geq 0$ and $T \in \mathcal{Z}(E)$.

Proof.

Using the property from the previous page, one direction is obvious.
Assume that $V(\mathcal{L}_r(E), T) \subseteq \mathbb{R}^+$. Since $T \in \mathcal{L}_r(E)$, $T = \mathcal{P}(T) + T_1$ where $\mathcal{P}(T) \in \mathcal{Z}(E)$ and $T_1 \perp I$. For any regular state, Φ, we have

$$\Phi(T) = \Phi(\mathcal{P}(T)) + \Phi(T_1).$$

Consider the regular states, Φ, such that $\Phi = 0$ on I^d. Then $\Phi(\mathcal{P}(T)) \geq 0$ for all such states which implies that $\mathcal{P}(T) \geq 0$.
Since $\Phi(\mathcal{P}(T)) \in \mathbb{R}$ for all regular states, we must also have $\Phi(T_1) \in \mathbb{R}$ for all regular states. If $T_1 \neq 0$ then there exists a regular state Φ such that $\Phi(T_1) \in \mathbb{C} \backslash \mathbb{R}$ which is a contradiction of $V(\mathcal{L}_r(E), T) \subseteq \mathbb{R}^+$. So $T_1 = 0$ and $T = \mathcal{P}(T)$. $\square$

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Positive Numerical Ranges

Definition

For $T \in \mathcal{L}_r(E)$ on a Dedekind complete Banach Lattice E, the positive spatial numerical range is defined to be

$$V_+(T) = \{f(Tx) : 0 \le x \in E, 0 \le f \in E^*, \|x\| = \|f\| = 1 = f(x)\}$$

Definition

For $T \in \mathcal{L}_r(E)$ on a Dedekind complete Banach Lattice E, the positive algebraic numerical range is defined to be

$$V_+(\mathcal{L}_r(E), T) = \{\Phi(T) : 0 \le \Phi \in \mathcal{L}_r(E)^*, \|\Phi\| = 1 = \Phi(I)\}.$$

Properties

Proposition

For $T \in \mathcal{L}_r(E)$ with $T \geq 0$, then $V_+(T)$ and $V_+(\mathcal{L}_r(E), T)$ are intervals in $[0, \infty)$.

Proposition

For $T \in \mathcal{L}_r(E)$ with $T \geq 0$, then

$$\sup\{\lambda : \lambda \in V_+(T)\} = \sup\{|\lambda| : \lambda \in V(T)\} = \sup\{\lambda : \lambda \in V_+(\mathcal{L}_r(E), T)\}.$$

Theorem

For $T \in \mathcal{L}_r(E)$ with $T \geq 0$, then

$$\inf\{\lambda : \lambda \in V_+(T)\} = \sup\{c : T \geq cI\}.$$

In particular, if $T \perp I$, then $0 \in \overline{V_+(T)}$.

Properties

Theorem

For $T \in \mathcal{L}_r(E)$ with $T \geq 0$ then

$$\overline{V_+(T)} = V_+(\mathcal{L}_r(E), T)$$

Proof.

From the previous slide we know that both sets are intervals in $[0, \infty)$ with the same supremum. We also have that $V_+(T) \subseteq V_+(\mathcal{L}_r(E), T)$, so consider a case where

$$\inf\{\lambda : \lambda \in V_+(\mathcal{L}_r(E), T)\} < \inf\{\lambda : \lambda \in V_+(T)\} = \delta.$$

By the previous theorem we have that $T - \delta I \geq 0$. However

$$V_+(\mathcal{L}_r(E), T - \delta I) \not\subseteq [0, \infty).$$

This yields a contradiction □

Duality

For $T \in \mathcal{L}(E)$, $V(T) \subseteq V(T^*)$ and $\overline{co}V(T) = \overline{co}V(T^*)$.

Theorem

Let E be a Dedekind complete Banach lattice over $\mathbb{C}$ with an order continuous norm. Then

$$V(\mathcal{L}_r(E), T) = V(\mathcal{L}_r(E^*), T^*)$$

In general, for $T \in \mathcal{L}_r(E)$ we have that $V_+(T) \subseteq V_+(T^*)$.

Theorem

For $T \in \mathcal{L}_r(E)$ with $T \geq 0$, we have that

$$\overline{V_+(T)} = \overline{V_+(T^*)}$$

Thank You