Galois connections between generating systems of sets and sequences

Rauni Lillemets

University of Tartu, Estonia

July 17, 2017
GREETINGS FROM ESTONIA
The aim of this talk

- Introduce the notions of generating systems of sets and sequences
- Study these notions
- Show that there is a deep relationship between these notions
The aim of this talk

- Introduce the notions of generating systems of sets and sequences
- Study these notions
The aim of this talk

- Introduce the notions of generating systems of sets and sequences
- Study these notions
- Show that there is a deep relationship between these notions
Historical background

- Operator ideals – introduced by A. Pietsch in 1978

Generating systems of sets and sequences – introduced by I. Stepaksi in 1980

Given two generating systems of sets, one obtains an operator ideal by considering all of the operators that map the sets of the first system to the sets of the second system.

Generating systems of sequences produce generating systems of sets.
Historical background

- Operator ideals – introduced by A. Pietsch in 1978
- Generating systems of sets and sequences – introduced by I. Stephani in 1980
Historical background

- Operator ideals – introduced by A. Pietsch in 1978
- Generating systems of sets and sequences – introduced by I. Stephani in 1980

- Given two generating systems of sets, one obtains an operator ideal by considering all of the operators that map the sets of the first system to the sets of the second system.
**Historical background**

- Operator ideals – introduced by A. Pietsch in 1978
- Generating systems of sets and sequences – introduced by I. Stephani in 1980

Given two generating systems of sets, one obtains an operator ideal by considering all of the operators that map the sets of the first system to the sets of the second system.

Generating systems of sequences produce generating systems of sets.
We always consider $X$ to be Banach space over the field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. 

Why?

- Simplifies notation
- Avoids leaving ZFC
- All results still hold in the general context
A remark

We always consider $X$ to be Banach space over the field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

- In order to produce operator ideals, Stephani considered systems of sets and sequences over all Banach spaces simultaneously.

Why?
- Simplifies notation
- Avoids leaving ZFC
- All results still hold in the general context
We always consider $X$ to be Banach space over the field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

- In order to produce operator ideals, Stephani considered systems of sets and sequences over all Banach spaces simultaneously.
- In this talk, we do the simplification of fixing a Banach space $X$ and we consider only systems of sequences in $X$ and systems of subsets of $X$. 

Why?
- Simplifies notation
- Avoids leaving ZFC
- All results still hold in the general context
A remark

We always consider $X$ to be a Banach space over the field $K$, where $K$ is either $\mathbb{R}$ or $\mathbb{C}$.

- In order to produce operator ideals, Stephani considered systems of sets and sequences over all Banach spaces simultaneously.
- In this talk, we do the simplification of fixing a Banach space $X$ and we consider only systems of sequences in $X$ and systems of subsets of $X$.

Why?
A remark

We always consider $X$ to be Banach space over the field $K$, where $K$ is either $\mathbb{R}$ or $\mathbb{C}$.

- In order to produce operator ideals, Stephani considered systems of sets and sequences over all Banach spaces simultaneously.
- In this talk, we do the simplification of fixing a Banach space $X$ and we consider only systems of sequences in $X$ and systems of subsets of $X$.

Why?

- Simplifies notation
We always consider $X$ to be Banach space over the field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

- In order to produce operator ideals, Stephani considered systems of sets and sequences over all Banach spaces simultaneously.
- In this talk, we do the simplification of fixing a Banach space $X$ and we consider only systems of sequences in $X$ and systems of subsets of $X$.

Why?

- Simplifies notation
- Avoids leaving ZFC
A remark

We always consider $X$ to be Banach space over the field $K$, where $K$ is either $\mathbb{R}$ or $\mathbb{C}$.

- In order to produce operator ideals, Stephani considered systems of sets and sequences over all Banach spaces simultaneously.
- In this talk, we do the simplification of fixing a Banach space $X$ and we consider only systems of sequences in $X$ and systems of subsets of $X$.

Why?

- Simplifies notation
- Avoids leaving ZFC
- All results still hold in the general context
Generating system of sets

Definition (Stephani, 1980)

A collection $G$ of bounded subsets of $X$ is a generating system of sets if

1. $G$ is closed under algebraic operations: if $G, H \in G$, then $\alpha G + \beta H \in G$ for any scalars $\alpha$ and $\beta$;
2. $G$ contains every bounded subset of every 1-dimensional subspace of $X$;
3. $G$ is closed under taking subsets: if $G \in G$ and $H \subseteq G$, then $H \in G$.

Denote the set of all generating systems of sets by $GSet$.

Denote by $B$ the system of all bounded sets in $X$.

Example: $K \in GSet$, $B \in GSet$.
Generating system of sets

**Definition (Stephani, 1980)**

A collection $G$ of bounded subsets of $X$ is a *generating system of sets* if

1. $G$ is closed under algebraic operations: if $G, H \in G$, then $\alpha G + \beta H \in G$ for any scalars $\alpha$ and $\beta$;

2. $G$ contains every bounded subset of every 1-dimensional subspace of $X$;

3. $G$ is closed under taking subsets: if $G \in G$ and $H \subseteq G$, then $H \in G$.

Denote the set of all generating systems of sets by $GSet$.

Denote by $B$ the system of all bounded sets in $X$.

Example: $K \in GSet$, $B \in GSet$. 

Rauni Lillemets

Galois connections between generating systems of sets and sequences
A collection $G$ of bounded subsets of $X$ is a *generating system of sets* if

1. $G$ is closed under algebraic operations: if $G, H \in G$, then $\alpha G + \beta H \in G$ for any scalars $\alpha$ and $\beta$;
2. $G$ contains every bounded subset of every 1-dimensional subspace of $X$;
Generating system of sets

Definition (Stephani, 1980)

A collection $G$ of bounded subsets of $X$ is a generating system of sets if

$(G_1)$ $G$ is closed under algebraic operations: if $G, H \in G$, then $\alpha G + \beta H \in G$ for any scalars $\alpha$ and $\beta$;

$(G_2)$ $G$ contains every bounded subset of every 1-dimensional subspace of $X$;

$(G_3)$ $G$ is closed under taking subsets: if $G \in G$ and $H \subseteq G$, then $H \in G$. 

Denote the set of all generating systems of sets by $G_{\text{Set}}$.

Denote by $B$ the system of all bounded sets in $X$.

Example: $K \in G_{\text{Set}}$, $B \in G_{\text{Set}}$. 

Rauni Lillemets
Galois connections between generating systems of sets and sequences
A collection $G$ of bounded subsets of $X$ is a generating system of sets if

1. $G$ is closed under algebraic operations: if $G, H \in G$, then $\alpha G + \beta H \in G$ for any scalars $\alpha$ and $\beta$;

2. $G$ contains every bounded subset of every 1-dimensional subspace of $X$;

3. $G$ is closed under taking subsets: if $G \in G$ and $H \subseteq G$, then $H \in G$.

Denote the set of all generating systems of sets by $GSet$. 

Example: $K \in GSet$, $B \in GSet$. 

Rauni Lillemets

Galois connections between generating systems of sets and sequences
Generating system of sets

Definition (Stephani, 1980)

A collection $G$ of bounded subsets of $X$ is a generating system of sets if

(G1) $G$ is closed under algebraic operations: if $G, H \in G$, then $\alpha G + \beta H \in G$ for any scalars $\alpha$ and $\beta$;

(G2) $G$ contains every bounded subset of every 1-dimensional subspace of $X$;

(G3) $G$ is closed under taking subsets: if $G \in G$ and $H \subseteq G$, then $H \in G$.

Denote the set of all generating systems of sets by $GSet$.
Denote by $B$ the system of all bounded sets in $X$. 
Generating system of sets

Definition (Stephani, 1980)

A collection $G$ of bounded subsets of $X$ is a generating system of sets if

\begin{align*}
(G_1) \quad & G \text{ is closed under algebraic operations: if } G, H \in G, \text{ then } \alpha G + \beta H \in G \text{ for any scalars } \alpha \text{ and } \beta; \\
(G_2) \quad & G \text{ contains every bounded subset of every 1-dimensional subspace of } X; \\
(G_3) \quad & G \text{ is closed under taking subsets: if } G \in G \text{ and } H \subseteq G, \text{ then } H \in G.
\end{align*}

Denote the set of all generating systems of sets by $GSet$.
Denote by $B$ the system of all bounded sets in $X$.
Example: $K \in GSet$, $B \in GSet$
Generating system of sequences

Denote by \( m \) the system of all bounded sequences in \( X \).
Denote by $\mathbf{m}$ the system of all bounded sequences in $X$.

**Definition (Stephani, 1980)**

A collection $\mathbf{g}$ of bounded sequences in $X$ is a *generating system of sequences* if

1. $\mathbf{g}$ is a linear subspace of $\mathbf{m}$;
2. every sequence which spans a 1-dimensional subspace of $X$ has a subsequence which is an element of $\mathbf{g}$;
3. $\mathbf{g}$ is closed under passing to subsequences.

Denote the set of all generating systems of sequences by $\mathbf{GSeq}$.

Examples:

$c \in \mathbf{GSeq}$,

$m \in \mathbf{GSeq}$.
Generating system of sequences

Denote by \( m \) the system of all bounded sequences in \( X \).

**Definition (Stephani, 1980)**

A collection \( g \) of bounded sequences in \( X \) is a *generating system of sequences* if

1. \((S_1)\) \( g \) is a linear subspace of \( m \);
Generating system of sequences

Denote by \( m \) the system of all bounded sequences in \( X \).

**Definition (Stephani, 1980)**

A collection \( g \) of bounded sequences in \( X \) is a *generating system of sequences* if

\( S_1 \) \( g \) is a linear subspace of \( m \);

\( S_2 \) every sequence which spans a 1-dimensional subspace of \( X \) has a subsequence which is an element of \( g \);
Generating system of sequences

Denote by $\mathbf{m}$ the system of all bounded sequences in $X$.

**Definition (Stephani, 1980)**

A collection $\mathbf{g}$ of bounded sequences in $X$ is a *generating system of sequences* if

$(S_1)$ $\mathbf{g}$ is a linear subspace of $\mathbf{m}$;

$(S_2)$ every sequence which spans a 1-dimensional subspace of $X$ has a subsequence which is an element of $\mathbf{g}$;

$(S_3)$ $\mathbf{g}$ is closed under passing to subsequences.
Denote by $\mathbf{m}$ the system of all bounded sequences in $X$.

**Definition (Stephani, 1980)**

A collection $\mathbf{g}$ of bounded sequences in $X$ is a *generating system of sequences* if

1. $(S_1)$ $\mathbf{g}$ is a linear subspace of $\mathbf{m}$;
2. $(S_2)$ every sequence which spans a 1-dimensional subspace of $X$ has a subsequence which is an element of $\mathbf{g}$;
3. $(S_3)$ $\mathbf{g}$ is closed under passing to subsequences.

Denote the set of all generating systems of sequences by $\mathcal{GSeq}$. 
Denote by $m$ the system of all bounded sequences in $X$.

**Definition (Stephani, 1980)**

A collection $g$ of bounded sequences in $X$ is a *generating system of sequences* if

- $(S_1)$ $g$ is a linear subspace of $m$;
- $(S_2)$ every sequence which spans a 1-dimensional subspace of $X$ has a subsequence which is an element of $g$;
- $(S_3)$ $g$ is closed under passing to subsequences.

Denote the set of all generating systems of sequences by $GSeq$. Examples: $c \in GSeq$, $m \in GSeq$
Let $g \in G\text{Seq}$.
Stephani defined a map $\Psi : G\text{Seq} \to G\text{Set}$ in the following way.

Definition (Stephani, 1980)
A set $G$ belongs to the system of sets $\Psi (g)$ iff every sequence $(x_k) \subset G$ contains a subsequence in the system of sequences $g$.

By definition, $\Psi (c) = K$.
Also, $\Psi (m) = B$. 

Rauni Lillemets
Galois connections between generating systems of sets and sequences
Let $g \in \text{GSeq}$.
Stephani defined a map $\Psi : \text{GSeq} \rightarrow \text{GSet}$ in the following way.

**Definition (Stephani, 1980)**

A set $G$ belongs to the system of sets $\Psi(g)$ iff every sequence $(x_k) \subset G$ contains a subsequence in the system of sequences $g$. 

By definition, $\Psi(c) = K$.
Also, $\Psi(m) = B$. 

---

Rauni Lillemets
Galois connections between generating systems of sets and sequences

8 / 17
Let \( g \in \text{GSeq} \).
Stephani defined a map \( \Psi : \text{GSeq} \rightarrow \text{GSet} \) in the following way.

**Definition (Stephani, 1980)**

A set \( G \) belongs to the system of sets \( \Psi (g) \) iff every sequence \( (x_k) \subset G \) contains a subsequence in the system of sequences \( g \).

By definition, \( \Psi (c) = K \).
Also, \( \Psi (m) = B \).
Let $G \in G\text{Set}$.
We define a map $\Phi : G\text{Set} \to G\text{Seq}$ in the following way.
Let $G \in \text{GSet}$.
We define a map $\Phi : \text{GSet} \rightarrow \text{GSeq}$ in the following way.

**Definition**

A sequence $(x_k)$ belongs to the system of sequences $\Phi(G)$ iff it is contained in $G$ for some $G \in \text{G}$. 
An alternative notion of relative compactness

A. Grothendieck proved the following result in his famous Memoir.

**Theorem (Grothendieck, 1955)**

A set $K \subset X$ is relatively compact if and only if there exists $x = (x_k) \in c_0(X)$ so that $K \subset \{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_1} \}$. 

Let us use the shorthands $\ell_{\infty}(X) := c_0(X)$ and $\ell_{\infty} := c_0$. 

**Definition (Ain, Lillemets, Oja, 2012)**

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. A set $K \subset X$ is said to be relatively $(p, r)$-compact, if there exists $x = (x_k) \in \ell_p(X)$ so that $K \subset \{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_r} \}$. 

Denote by $K((p, r))$ the set of all relatively $(p, r)$-compact sets in $X$. 

Observe that $K((\infty, 1)) = K$. 

Rauni Lillemets
Galois connections between generating systems of sets and sequences
An alternative notion of relative compactness

A. Grothendieck proved the following result in his famous Memoir.

**Theorem (Grothendieck, 1955)**

A set \( K \subset X \) is relatively compact if and only if there exists \( x = (x_k) \in c_0(X) \) so that \( K \subset \{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_1} \} \).

Let us use the shorthands \( \ell_\infty(X) := c_0(X) \) and \( \ell_\infty := c_0 \).

**Definition (Ain, Lillemets, Oja, 2012)**

Let \( 1 \leq p \leq \infty \) and \( 1 \leq r \leq p^* \).

A set \( K \subset X \) is said to be \textit{relatively \((p, r)\)-compact}, if there exists \( x = (x_k) \in \ell_p(X) \) so that \( K \subset \{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in B_{\ell_r} \} \).
An alternative notion of relative compactness

A. Grothendieck proved the following result in his famous Memoir.

**Theorem (Grothendieck, 1955)**

A set \( K \subset X \) is relatively compact if and only if there exists \( x = (x_k) \in c_0(X) \) so that \( K \subset \{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in \mathcal{B}_{\ell_1} \} \).

Let us use the shorthands \( \ell_\infty(X) := c_0(X) \) and \( \ell_\infty := c_0 \).

**Definition (Ain, Lillemets, Oja, 2012)**

Let \( 1 \leq p \leq \infty \) and \( 1 \leq r \leq p^* \).

A set \( K \subset X \) is said to be relatively \((p, r)\)-compact, if there exists \( x = (x_k) \in \ell_p(X) \) so that \( K \subset \{ \sum_{k=1}^{\infty} \alpha_k x_k \mid (\alpha_k) \in \mathcal{B}_{\ell_r} \} \).

Denote by \( K_{(p, r)} \) the set of all relatively \((p, r)\)-compact sets in \( X \).

Observe that \( K_{(\infty, 1)} = K \).
In 1980, Stephani gave an answer to the following question.

**Question (Stephani, 1980)**

Which operator ideals can be represented as operators mapping bounded sets to sets of a system $G$, where $G \in GSet$?

Answer: all surjective operator ideals.
A question

In 1980, Stephani gave an answer to the following question.

**Question (Stephani, 1980)**

Which operator ideals can be represented as operators mapping bounded sets to sets of a system $G$, where $G \in \text{GSet}$?

**Answer:** all *surjective operator ideals*. 
In 1980, Stephani gave an answer to the following question.

**Question (Stephani, 1980)**

Which operator ideals can be represented as operators mapping bounded sets to sets of a system $G$, where $G \in \text{GSet}$?

**Answer:** all *surjective* operator ideals.

**Definition**

A system $G \in \text{GSet}$ is said to be *sequentially generatable* if there exists $g \in \text{GSeq}$ so that $G = \Psi(g)$.
A question

In 1980, Stephani gave an answer to the following question.

**Question (Stephani, 1980)**

Which operator ideals can be represented as operators mapping bounded sets to sets of a system \( G \), where \( G \in GSet \)?

**Answer:** all *surjective operator ideals*.

**Definition**

A system \( G \in GSet \) is said to be *sequentially generatable* if there exists \( g \in GSeq \) so that \( G = \Psi(g) \).

**Questions**

Which generating systems of sets are sequentially generatable? For example, is \( K_{(p,r)} \) sequentially generatable?
Galois connection

Let $A$ and $B$ be ordered sets.

**Definition**

A pair $(R, S)$ of maps $R: A \rightarrow B$ and $S: B \rightarrow A$ is called a Galois connection if $R(a) \leq b \iff a \leq S(b)$.

**Lemma**

Assume that the pair $(R, S)$ is a Galois connection between ordered sets $A$ and $B$. Let $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then

1. $a \leq SR(a)$ and $RS(b) \leq b$;
2. $a_1 \leq a_2 \Rightarrow R(a_1) \leq R(a_2)$ and $b_1 \leq b_2 \Rightarrow S(b_1) \leq S(b_2)$;
3. $R(a) = RS(a)$ and $S(b) = SRS(b)$.

For a given $a \in A$ there exists $b \in B$ such that $S(b) = a$ if and only if $a = SR(a)$. 

Rauni Lillemets
Galois connections between generating systems of sets and sequences
Galois connection

Let $A$ and $B$ be ordered sets.

**Definition**

A pair $(R, S)$ of maps $R : A \rightarrow B$ and $S : B \rightarrow A$ is called a *Galois connection* if $R(a) \leq b \iff a \leq S(b)$. 
# Galois connection

Let $A$ and $B$ be ordered sets.

## Definition

A pair $(R, S)$ of maps $R: A \to B$ and $S: B \to A$ is called a *Galois connection* if $R(a) \leq b \iff a \leq S(b)$.

## Lemma

Assume that the pair $(R, S)$ is a Galois connection between ordered sets $A$ and $B$. Let $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then

1. $a \leq SR(a)$ and $RS(b) \leq b$;
2. $a_1 \leq a_2 \Rightarrow R(a_1) \leq R(a_2)$ and $b_1 \leq b_2 \Rightarrow S(b_1) \leq S(b_2)$;
3. $R(a) = RSR(a)$ and $S(b) = SRS(b)$.
Let $A$ and $B$ be ordered sets.

**Definition**

A pair $(R, S)$ of maps $R: A \to B$ and $S: B \to A$ is called a *Galois connection* if $R(a) \leq b \iff a \leq S(b)$.

**Lemma**

Assume that the pair $(R, S)$ is a Galois connection between ordered sets $A$ and $B$. Let $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Then

1. $a \leq SR(a)$ and $RS(b) \leq b$;
2. $a_1 \leq a_2 \Rightarrow R(a_1) \leq R(a_2)$ and $b_1 \leq b_2 \Rightarrow S(b_1) \leq S(b_2)$;
3. $R(a) = RSR(a)$ and $S(b) = SRS(b)$.

For a given $a \in A$ there exists $b \in B$ such that $S(b) = a$ if and only if $a = SR(a)$. 
The set $G_{Set}$ is ordered with respect to inclusion $\subseteq$. Note that $G \subseteq B$ for every $G \in G_{Set}$.
Defining orders on systems of sets and sequences

The set $GSet$ is ordered with respect to inclusion $\subseteq$. Note that $G \subseteq B$ for every $G \in GSet$.

**Definition (Stephani, 1980)**

Let $g, h \in GSeq$. The system $h$ is said to dominate the system $g$, written $g \preceq h$, if every sequence from $g$ has a subsequence in $h$.

**Proposition**

The relation $\preceq$ is a preorder on the class $GSeq$. But it is not an order (since it does not satisfy the axiom of antisymmetry).

**Definition**

Given $g, h \in GSeq$, we write $g \sim h$ if $g \preceq h$ and $h \preceq g$.

Relation $\sim$ is an equivalence relation on $GSeq$. Preorder $\preceq$ on $GSeq$ induces an order on $GSeq/\sim$ via $[g] \leq [h]$ whenever $g \preceq h$. 
Defining orders on systems of sets and sequences

The set $\mathbb{GSet}$ is ordered with respect to inclusion $\subseteq$. Note that $G \subseteq B$ for every $G \in \mathbb{GSet}$.

**Definition (Stephani, 1980)**

Let $g, h \in \mathbb{GSeq}$. The system $h$ is said to dominate the system $g$, written $g \preceq h$, if every sequence from $g$ has a subsequence in $h$.

**Proposition**

The relation $\preceq$ is a preorder on the class $\mathbb{GSeq}$. But it is not an order (since it does not satify the axiom of antisymmetry).
Defining orders on systems of sets and sequences

The set $G_{\text{Set}}$ is ordered with respect to inclusion $\subseteq$. Note that $G \subseteq B$ for every $G \in G_{\text{Set}}$.

**Definition (Stephani, 1980)**

Let $g, h \in G_{\text{Seq}}$. The system $h$ is said to dominate the system $g$, written $g \preceq h$, if every sequence from $g$ has a subsequence in $h$.

**Proposition**

The relation $\preceq$ is a preorder on the class $G_{\text{Seq}}$. But it is not an order (since it does not satisfy the axiom of antisymmetry).

**Definition**

Given $g, h \in G_{\text{Seq}}$, we write $g \sim h$ if $g \preceq h$ and $h \preceq g$. 

Rauni Lillemets

Galois connections between generating systems of sets and sequences
Defining orders on systems of sets and sequences

The set $GSet$ is ordered with respect to inclusion $\subseteq$. Note that $G \subseteq B$ for every $G \in GSet$.

**Definition (Stephani, 1980)**

Let $g, h \in GSeq$. The system $h$ is said to dominate the system $g$, written $g \preceq h$, if every sequence from $g$ has a subsequence in $h$.

**Proposition**

The relation $\preceq$ is a preorder on the class $GSeq$. But it is not an order (since it does not satisfy the axiom of antisymmetry).

**Definition**

Given $g, h \in GSeq$, we write $g \sim h$ if $g \preceq h$ and $h \preceq g$.

Relation $\sim$ is an equivalence relation on $GSeq$. Preorder $\preceq$ on $GSeq$ induces an order on $GSeq/\sim$ via $[g] \leq [h]$ whenever $g \preceq h$. 

Rauni Lillemets
Galois connections between generating systems of sets and sequences
Maps between $GSet$ and $GSeq/\sim$

**Proposition**

Let $g, h \in GSeq$. Then $g \sim h$ iff $\Psi(g) = \Psi(h)$.
Maps between $\text{GSet}$ and $\text{GSeq}/\sim$

**Proposition**

Let $g, h \in \text{GSeq}$. Then $g \sim h$ iff $\Psi (g) = \Psi (h)$.

**Definition**

Define the operator $\phi : \text{GSet} \rightarrow \text{GSeq}/\sim$ by

$$\phi (G) := \left[ \Phi (G) \right], \text{ where } G \in \text{GSet}.$$ 

Define the operator $\psi : \text{GSeq}/\sim \rightarrow \text{GSet}$ by

$$\psi ([g]) := \Psi (g), \text{ where } [g] \in \text{GSeq}/\sim.$$
Maps between $\text{GSet}$ and $\text{GSeq}/\sim$

**Proposition**

Let $g, h \in \text{GSeq}$. Then $g \sim h$ iff $\Psi(g) = \Psi(h)$.

**Definition**

Define the operator $\phi : \text{GSet} \rightarrow \text{GSeq}/\sim$ by

$$\phi(G) := [\Phi(G)], \text{ where } G \in \text{GSet}.$$  

Define the operator $\psi : \text{GSeq}/\sim \rightarrow \text{GSet}$ by

$$\psi([g]) := \Psi(g), \text{ where } [g] \in \text{GSeq}/\sim.$$  

For example, $\psi([c]) = \Psi(c) = K$. 
Theorem

The pair \((\phi, \psi)\) is a Galois connection between \(G\text{Set}\) and \(G\text{Seq}/\sim\).
Theorem
The pair \((\phi, \psi)\) is a Galois connection between \(\text{GSet}\) and \(\text{GSeq}/\sim\).

Corollary
Let \(G \in \text{GSet}\). Then \(G\) is sequentially generatable iff
\[\psi(\phi(G)) = \Psi(\Phi(G)) = G.\]
Theorem
The pair \((\phi, \psi)\) is a Galois connection between \(G\text{Set}\) and \(G\text{Seq}/\sim\).

Corollary
Let \(G \in G\text{Set}\). Then \(G\) is sequentially generatable iff 
\[
\psi(\phi(G)) = \psi(\Phi(G)) = G.
\]

Proposition
Let \(1 \leq p \leq \infty\) and let \(1 \leq r \leq p^*\). Then 
\[
\psi(\Phi(K_{p,r})) = K.
\]
Galois connection between $GSet$ and $GSeq/\sim$

**Theorem**
The pair $(\phi, \psi)$ is a Galois connection between $GSet$ and $GSeq/\sim$.

**Corollary**
Let $G \in GSet$. Then $G$ is sequentially generatable iff $\psi(\phi(G)) = \psi(\Phi(G)) = G$.

**Proposition**
Let $1 \leq p \leq \infty$ and let $1 \leq r \leq p^*$. Then $\Psi(\Phi(K(p,r))) = K$.

**Theorem (Delgado, Piñeiro, Serrano, 2010)**
Let $1 \leq p < \infty$. If $X$ is infinite dimensional, then there exists a compact set $K \subseteq X$ such that $K \not\in K(p,r)$. 

Rauni Lillemets  Galois connections between generating systems of sets and sequences  15 / 17
To conclude:

- The notion of a generating system of sets is a useful tool for generating new operator ideals.
- There is an intimate relationship between generating systems of sets and sequences.
- Not all generating systems of sets can be created from systems of sequences—but at least we have a criterion for deciding which of them can be.

This talk was based on a paper "Galois connections between generating systems of sets and sequences", which was published (online) in 2016 in the journal Positivity.
To conclude:

- The notion of a generating system of sets is a useful tool for generating new operator ideals.
To conclude:

- The notion of a generating system of sets is a useful tool for generating new operator ideals.
- There is an intimate relationship between generating systems of sets and sequences.
To conclude:

- The notion of a generating system of sets is a useful tool for generating new operator ideals.
- There is an intimate relationship between generating systems of sets and sequences.
- Not all generating systems of sets can be created from systems of sequences – but at least we have a criterion for deciding which of them can be.
To conclude:

- The notion of a generating system of sets is a useful tool for generating new operator ideals.
- There is an intimate relationship between generating systems of sets and sequences.
- Not all generating systems of sets can be created from systems of sequences – but at least we have a criterion for deciding which of them can be.

This talk was based on a paper “Galois connections between generating systems of sets and sequences”, which was published (online) in 2016 in the journal Positivity.
Thank you for listening!