The Bishop-Phelps-Bollobás property on closed bounded convex sets

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Bishop-Phelps Theorem

Theorem (E. Bishop and R.R. Phelps (1961))

Let C be a closed bounded convex set in a real Banach space X. Then the set of linear functionals that attain their maximum on C is dense in X^*.

In particular, the set of all norm-attaining linear functionals on a Banach space X is dense in the dual space X^*.

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V. Lomonosov (2000)

The Bishop-Phelps theorem cannot be extended to general complex Banach spaces by constructing a closed bounded convex set with no support points.

Norm Attaining Mappings

$X, \ Y =$ Real or Complex Banach Space
Let S_X and B_X be the unit sphere and closed unit ball of X, respectively.

$T \in L(X, Y)$ attains its norm if there is $x_0 \in S_X$ such that
$\|T(x_0)\| = \|T\|$.

$NA(L(X, Y)) =$ Set of all norm-attaining linear mappings from X into Y.

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Question.

Is the set $NA(L(X, Y))$ dense in $L(X, Y)$?

Norm Attaining Mappings

(Lindenstrauss, 1963)
Counterexamlpe: X= c_0, Y = Equivalently Renormed Space c_0 to be Strictly Convex.

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A Banach space X has property (A) if $NA(L(X, Y))$ is dense in $L(X, Y)$ for every Banach space Y.

A Banach space Y has property (B) if $NA(L(X, Y))$ is dense in $L(X, Y)$ for every Banach space X.

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Question. (The Most Irritating Open Problem)

Does the 2-dimensional Euclidean space $\mathbb{R}^2$ have property (B) ?

Theorem (J. Bourgain (1977))

A Banach space X has the Radon-Nikodym Property if and only if every Banach space isomorphic to X has property (A).

Examples with RNP : (1) Reflexive spaces (2) Separable Duals (3) WCG Duals (4) Locally Uniformly Convex Space (5) $l_1(I)$, I, any set

Nonlinear Version of Bourgain's Result

Theorem (C. Stegall (1978))

Let X be a Banach space with RNP, D be a bounded closed convex subset of X and $f : D \to \mathbb{R}$ be an upper semicontinuous bounded above function. Then for $\epsilon > 0$, there exists $x^* \in X^*$ such that $\|x^*\| < \epsilon$ and $f + x^*$, $f + |x^*|$ strongly expose D.

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Applying this result to a vector-valued case, he showed the following.

Theorem (C. Stegall (1978))

Let X be a Banach space with RNP, D be a bounded closed convex subset of X, and Y be a Banach space. Suppose that $\varphi : D \to Y$ is a uniformly bounded function such that the function $x \to \|\varphi(x)\|$ is upper semicontinuous. Then, for $\delta > 0$, there exist $T : X \to Y$ a bounded linear operator of rank one, $\|T\| < \delta$ such that $\varphi + T$ attains its supremum in norm on D and does so at most two points

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The Bishop-Phelps theorem (Bishop and Phelps, 1961)

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Theorem (Bollobás, 1970)

For $\epsilon > 0$, if $x \in B_X$ and $x^* \in S_{X^*}$ satisfy $|1 - x^*(x)| < \frac{\epsilon^2}{4}$, then there are $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.

Bronsted-Rockafellar Theorem

Theorem (Bronstead-Rockafellar Theorem, PAMS, 1965)

Suppose that f is a convex proper lower semicontinuous function on the Banach space X. The given any point $x_0 \in dom(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exist $x \in dom(f)$ and $x^* \in X^*$ such that

$$x^* \in \partial(f), \ \|x - x_0\| \leq \frac{\epsilon}{\lambda}, \ \textit{and} \ \|x^* - x_0^*\| \leq \lambda.$$

In particular, the domain of ∂f is dense in $dom(f)$.

Definition (Acoasta, Aron, García and Maestre, JFA 2008)

We say that the couple (X, Y) satisfies the Bishop-Phelps-Bollobás property for operators (BPBp for short), if given $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that for $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions :

$$\|Su_0\| = 1, \|x_0 - u_0\| < \epsilon \text{ and } \|T - S\| < \epsilon.$$

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They showed that if a Banach space Y has property (β), then the couple (X, Y) has the *BPBp* for every Banach space X.

The Bishop-Phelps-Bollobás version of (Lindenstrauss) properties A and B

Recall that J. Lindenstrauss introduced the following two properties.

A Banach space X is said to have *Lindenstrauss property A* if $\overline{NA(X,Z)} = L(X,Z)$ for every Banach space Z.

A Banach space Y is said to have *Lindenstrauss property B* if $\overline{NA(Z,Y)} = L(Z,Y)$ for every Banach space Z.

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Definition

Let X and Y be Banach spaces.
We say that X is a universal BPB domain space if for every Banach space Z, the pair (X,Z) has the BPBp.

We say that Y is a universal BPB range space if for every Banach space Z, the pair (Z,Y) has the BPBp.

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Theorem (R. Aron, C, S.K. Kim, H.J. Lee and M. Martin, Transactions AMS 2015)

Every Banach space isomorphic to X is a universal BPB domain space if and only if X is the basic field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Operators from $\ell_p(c_0) \to Y$

Theorem (Acoasta, Aron, García and Maestre, JFA 2008)

The couple (ℓ_1, Y) satisfies the Bishop-Phelps-Bollobás property for operators if and only if Y has the AHSP.

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The following Banach spaces have the *AHSP*:

(a) a finite dimensional space, (b)a real or complex space $L_1(\mu)$ for a σ-finite measure μ,
(c)a real or complex space $C(K)$ for a compact Hausdorff space K, and
(d) a uniformly convex space.

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Theorem (AGGM (TAMS, 2012), H.J Lee and S. K. Kim (Canadian J. Math. 2013))

Let X be a uniformly convex Banach space. Then the couple (X, Y) has the BPBp for every Banach space Y.
More precisely, given $0 < \epsilon < 1$, let $0 < \eta < \frac{\epsilon}{8+2\epsilon}\delta(\epsilon)$. If $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|Tx_0\| > 1 - \eta,$$

then there exist $S \in S_{\mathcal{L}(X,Y)}$ and $u_0 \in S_X$ such that $\|Su_0\| = 1$, $\|S - T\| < \epsilon$ and $\|x_0 - u_0\| < \epsilon$.

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Answer : Yes [Sun Kwang Kim, Israel J. Math. 2013].

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Theorem (Aron, Cascales, Kozhushkina, PAMS 2011)

Let L be a locally compact space. Then $(X, C_0(L))$ has the BPBp if X is Asplund.
In particular, $(c_0, C_0(L))$ has the BPBp.

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Theorem (Cascales, Guirao, and Kadets, Advances in Math. 2012)

Let A be a uniform algebra. Then (X, A) has the BPBp if X is Asplund.
In particular, (c_0, A) has the BPBp.

Definition Let X and Y be Banach spaces. Let D be a bounded convex subset of X. We say that (X, Y) has the BPB_p on D if, for every $\varepsilon > 0$, there is $\eta_D(\varepsilon) > 0$ such that for every $T \in L(X, Y)$, $\|T\|_D = 1$ and for every $x \in D$ satisfying

$$\|Tx\| > 1 - \eta_D(\varepsilon),$$

there exist $S \in L(X, Y)$ and $z \in D$ such that

$$\|Sz\| = 1 = \|S\|_D, \quad \|x - z\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

$$\|T\|_D = \sup\{\|Tx\| : x \in D\}$$

We can see that BPBp holds for bounded linear functionals on arbitrary bounded convex sets of a real Banach space X.

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Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be proper lower semicontinuous and bounded below function on a real Banach space X.

Then, given $\varepsilon > 0$ and $\delta > 0$, there exists $x_1 \in X$ such that $f(x_1) < f(x) + \varepsilon \|x - x_1\|$ for every $x \in X$ with $x \neq x_1$. Moreover, if $f(x_0) < b + \delta/2$, where $b = \inf\{f(x) : x \in X\}$, then x_1 can be chosen so that $\|x_0 - x_1\| < \delta/\varepsilon$.

Theorem (Cho/C, JLMS 2016)

Let D be a bounded convex closed subset of a real Banach space X. Given $\varepsilon > 0$ and $\delta > 0$, if $f \in X^*$ and $x_0 \in D$ such that

$$f(x_0) > \sup\{f(x) : x \in D\} - \frac{\delta}{2},$$

then there exist $g \in X^*$ and $x_1 \in D$ satisfying $g(x_1) = \sup\{g(x) : x \in D\}$, $\|f - g\| \leq \varepsilon$ and $\|x_1 - x_0\| \leq \delta/\varepsilon$.

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We can also obtain the following theorem for a bounded linear functional, which is analogous to Stegall's nonlinear form.

Theorem Let D be a bounded convex set in a real Banach space X. Given $0 < \varepsilon < 1/4$ and $f \in X^*$, there exist $x^* \in X^*$ and $x_0 \in D$ such that both $f + x^*$ and $f + |x^*|$ attain their suprema simultaneously at x_0 and $\|x^*\| < \varepsilon$. Moreover, $(f + x^*)(x_0) = (f + |x^*|)(x_0)$.

Theorem Let D be a bounded convex set in a real Banach space X. Given $0<\varepsilon<1/4$ and $f\in X^*$, there exist $x^*\in X^*$ and $x_0\in D$ such that both $f+x^*$ and $f+|x^*|$ attain their suprema simultaneously at x_0 and $\|x^*\|<\varepsilon$. Moreover, $(f+x^*)(x_0)=(f+|x^*|)(x_0)$.

Sketch of proof

Assume $D\subseteq B_X$ and $\|f\|_D=1$.

B-P theorem $\Longrightarrow$ $\exists\; x_0^*\in X^*$ s.t. $\|x^*\|<\frac{\varepsilon}{2}$ and $f+x^*$ attains its supremum at $x_0\in D$.

If $f(x_0)+x^*(x_0)\geqslant f(x)+|x^*(x)|$, $\forall x\in D$, we are done.

Otherwise, $\exists\, y \in D$ s.t. $f(y) + |x^*(y)| > f(x_0) + x^*(x_0)$.
Clearly, $x^*(y) < 0$, and

$$f(y) - x^*(y) > f(x_0) + x^*(x_0).$$

Otherwise, $\exists\, y \in D$ s.t. $f(y) + |x^*(y)| > f(x_0) + x^*(x_0)$.

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Let $S = \sup_{x \in D} (f(x) - x^*(x))$ and

$\alpha = S - (f(x_0) + x^*(x_0)) < (1 + \frac{\varepsilon}{2}) - (1 - \frac{\varepsilon}{2}) = \varepsilon.$

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Choose $y_0 \in D$ so that $f(y_0) - x^*(y_0) > S - \frac{\alpha^2 \varepsilon^2}{2}$.
$\exists\, x_1^*$ s.t. $\underline{(f - x^*)} + x_1^*$ attains its supremum at $z_0 \in D$, $\|x_1^*\| \le \alpha\varepsilon$ and $\|y_0 - z_0\| \le \alpha\varepsilon$.

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$\exists\, x_1^*$ s.t. $(f - x^*) + x_1^*$ attains its supremum at $z_0 \in D$, $\|x_1^*\| \leq \alpha\varepsilon$ and $\|y_0 - z_0\| \leq \alpha\varepsilon$.
Set $x_2^* = -x^* + x_1^*$. we can check that $\|x_2^*\| < \varepsilon$ & $f + |x_2^*|$ attains its supremum at z_0 on D.

$f(z_0) + x_2^*(z_0) =$

Further, we can show that, for a bounded closed convex set D, the set $\{f : |f| \text{ attains its supremum on } D\}$ is dense in X^*.

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<u>Theorem</u> Let D be a bounded closed convex set in a real Banach space X. Given $f \in X^*$ and $\varepsilon > 0$, there exists $x^* \in X^*$ such that $|f + x^*|$ attains its supremum on D and $\|x^*\| \leq \varepsilon$. Moreover, if D is symmetric and $f(x_0) > \|f\|_D - \delta/2$ for some $x_0 \in D$ and $\delta > 0$, then x^* and $x_1 \in D$ can be chosen so that

$\|x^*\| \leq \varepsilon$, $\|x_0 - x_1\| \leq \delta/\varepsilon$ and $|f + x^*|$ attains its supremum at x_1 on D

Vector-valued case

(ℓ_2^2, Y) has BPBp on $B_{\ell_2^2}$ for every Banach space Y, but there is a Banach space Z such that (ℓ_2^2, Z) fails to have BPBp on $D=\{|x|+|y|\leq 1\}$.

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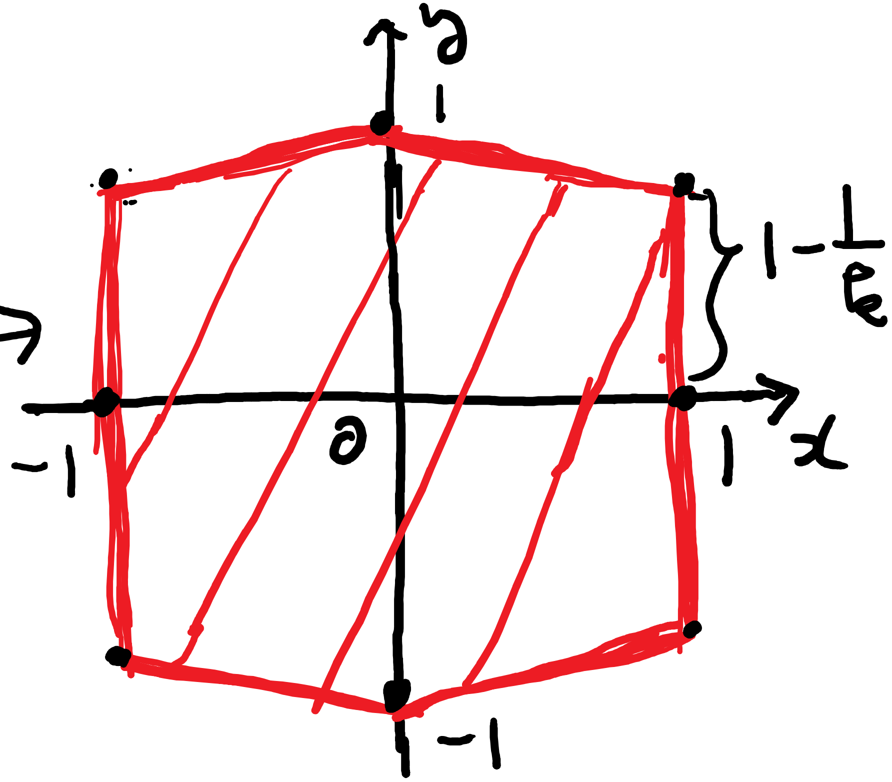
$$Z = \left[\bigoplus_{k=1}^{\infty} Z_k\right]_{\ell_\infty}$$

B_{Z_k}



Positive Results

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Recall The modulus of convexity

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}$$

For a bounded closed absorbing convex set D, define $\delta_D(\varepsilon)$ for $0 < \varepsilon < 1$ by

$$\delta_D(\varepsilon) = \inf\left\{\frac{P_D(x)}{2} + \frac{P_D(y)}{2} - P_D\left(\frac{x+y}{2}\right) : x, y \in D, P_D(x-y) \geq \varepsilon\right\}$$

Theorem. Let X and Y be (real or complex) Banach spaces and D be a bounded closed absorbing convex subset of B_X such that

$$\delta_D(\varepsilon) > 0 \text{ for every } 0 < \varepsilon < \tfrac{1}{2}.$$

If $T \in S_{L(X,Y)}$ and $x_1 \in D$ satisfy

$$\|Tx_1\| > \|T\|_D - \varepsilon^3 \delta_D(\varepsilon),$$

for sufficiently small ε relatively to $\|T\|_D$, then there exist $S \in L(X,Y)$ and $z \in D$ such that $\|Sz\| = \|S\|_D$, $\|S-T\| < 4\varepsilon^2/(1-\varepsilon)$,

and $\|x_1 - z\| \leq \rho_D(x_1 - z) < \varepsilon/(1-\varepsilon)$.

Aron, Cascales & Kozhushkina, PAMS 2011

BPBp holds on B_X for an Asplund operator from X into $C_0(L)$.

Aron, Cascales & Kozhushkina, PAMS 2011

BPBp holds on B_X for an Asplund operator from X into $C_0(L)$.

<u>Theorem</u> For a symmetric bounded closed convex subset $D \subseteq B_X$ and locally compact Hausdorff space L, let $T : X \to C_0(L)$ be an Asplund operator with $\|T\|_D = 1$ & $\|T\| = M \geq 1$. Given $0 < \varepsilon < M/2$, if $x_0 \in D$ satisfies that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{4M}$, then there exist an Asplund operator S and $u_0 \in D$ s.t. $\|S\|_D = 1 = \|Su_0\|$, $\|x_0 - u_0\| < \varepsilon$ and $\|T - S\| < 4\varepsilon$.

Thank you

Thank you for your attention !!