Lp-Spaces with respect to conditional expectation on Riesz spaces

Youssef. AZOUZI

University of Carthage,
Tunisia

Positivity IX
July 2017, University of Alberta, Edmonton
Some notations
We consider
a Dedekind complete Riesz space $E$
with weak order unit $e$, and
a conditional expectation $T$.
Here $T : E \rightarrow E$ satisfies the following conditions

1. positive projection,
2. order continuous
3. $Te = e$,
4. $T$ is strictly positive (i.e., $Tx > 0$ whenever $x > 0$),
5. $R(T)$ is a Dedekind complete Riesz subspace of $E$. 
The sup-completion of a Riesz space
This notion is introduced by Donner in 1982
$E_s$ plays the same role for $E$ as $\mathbb{R}_\infty : \mathbb{R} \cup \{\infty\}$ does for $\mathbb{R}$.
We recall that $E_s$ is a Dedekind complete lattice cone which satisfies the following conditions.

1. $E$ is an ordered subset of $E_s$,
2. $E_s$ has a biggest element,
3. If $x \in E_s$ then $x = \sup \{y \in E : y \leq x\}$,
4. if $y \leq x$ with $x \in E$ and $y \in E_s$ then $y \in E$.

Examples
1. If $E = \mathbb{R}$ then $E_s = \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$
2. More generally if $E = \mathbb{R}^n$ then $E_s = \mathbb{R}_\infty^n$.
3. If $E = L^p$ then $E_s = \{f$ measurable: $f \geq g$ for some $g \in L^p\}$.
Functional Calculus

If \( f \) is a real function and \( x \in E \), what is the meaning of \( f(x) \)?
We will use two kinds of functional calculus.

1. In the sense of Buskes, de Pagter, and von Rooij (1991).
   For \( f \in \mathbb{R}^\mathbb{R} \), the equality \( b = f(x) \) in \( E \) means that there exists a Riesz subspace \( V \) of \( E \) such that
   - \( b, x \in V \);
   - \( H(V) \) separates the points of \( V \);
   - \( \omega(b) = f(\omega(x)) \) for all \( \omega \in H(V) \).

2. In the sense of Grobler
   Here we use the Daniell Integral on Riesz spaces.
First kind

- Let $\mathcal{H}(E)$ be the set of all Riesz homomorphisms on $E$.
- The Riesz subspace of $E$ generated by a subset $A \subset E$ is denoted by $\langle A \rangle_E$.
- If $A$ is finite and $V = \langle A \rangle_E$ then $\mathcal{H}(V)$ *separates the points* of $V$.
- If such a $b$ exists, it is unique.
- If, in addition, $E$ is an $f$-algebra, we use $\mathcal{H}_m(V)$ rather than $\mathcal{H}(V)$. Here, $\mathcal{H}_m(V)$ is the subset of $\mathcal{H}(V)$ of all multiplicative.
Second kind

For $x \in E$ and $t \in \mathbb{R}$, let

\[ p_t = e - P_{(x-te)+e}, \quad t \in \mathbb{R} \]

\[ L = \text{span} \left\{ \chi_{(a,b]} \right\} \subset \mathbb{R}^\mathbb{R}. \]

Define $f(x)$ by

1. $f(x) = p_b - p_a$ if $f = \chi_{(a,b]}$
2. the definition is extended via a linearity process on $L$.
3. If $f_n \in L$, $f_n \geq 0$ and $f_n \uparrow f$ in $\mathbb{R}^\mathbb{R}$, we put $f(x) = \sup f_n(x) \in E_s$, (This is well defined)
4. If $f^+(x)$ and $f^-(x) \exists$ and $f^-(x) \in E$ we put $f(x) = f^+(x) - f^-(x) \in E_s$. 

Y. Azouzi (IPEST)
Some results

- If $f_n \rightarrow f$ uniformly and $f_n^D(x)$ and $f^D(x)$ exist in $E$. Then $f_n^D(x) \rightarrow f^D(x)$ in order in $E$.
- If $f$ is continuous and $f \circ g$ is well-defined and $g^D(x) \in E$. Then
  \[(f \circ g)^D(x) = f^D \left(g^D(x)\right) \in E_s.\]
- $f^D$ and $f^H$ coincide on $l_e$.
- If $f$ is increasing and continuous and $f^D(x) \in E$ then $f^D(x \land ne) \uparrow f^D(x)$ in $E$ and if $f(x \lor -ke) \in E$ for some $k \in \mathbb{N}$ then $f^D(x \lor -ne) \downarrow f^D(x)$ in $E$.
- If $f$ is continuous and $f^H(x)$ and $f^H(y)$ exist then
  1. $f$ increasing $\implies f^H(x \land y) = f^H(x) \land f^H(y)$.
  2. $f(0) = 0$ and $x \perp y \implies f^H(x + y) = f^H(x) + f^H(y)$.
Some results

- If $f$ is increasing and continuous and $f^D(x)$ and $f^D(y)$ exist in $E$ then
  1. $f^D(x \wedge y) = f^D(x) \wedge f^D(y)$.
  2. If $x \perp y$ then $f^D(x + y) = f^D(x) + f^D(y) - f(0) e$.

- Assume that $E$ is in addition an $f$-algebra with $e$ as multiplicative identity.
  Let $x \in E^+$ and $f, g \in \mathbb{R}^{\mathbb{R}}$ be continuous functions on $\mathbb{R}^+$ of bounded variation on each closed interval $[0, a]$ with $a \in (0, \infty)$. If $f^D(x), g^D(x), (fg)^D(x)$ exist in $E$ then $(fg)^D(x) = f^D(x)g^D(x)$.

- Let $f$ be a convex increasing real-valued function on $[0, \infty)$ and $(x_\alpha)$ be an increasing net in $E^+$ with $x = \sup x_\alpha$. If $f(x) \in E$. then $f(x_\alpha) \uparrow f(x)$. 
Convex functions

A function \( f : C \longrightarrow E \) is said to be

- convex if

\[
  f(tx + (1-t)y) \leq tf(x) + (1-t)f(t)
\]

for all \( x, y \in C, t \in [0,1] \).
Convex functions

A function $f : C \rightarrow E$ is said to be

- convex if
  $$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(t)$$
  for all $x, y \in C$, $t \in [0, 1]$.

- positively homogeneous if ($C$ is a cone and)
  $$f(tx) = tf(x) \quad \text{for all } x \in C \text{ and } t \in [0, \infty)$$
Convex functions

A function $f : C \rightarrow E$ is said to be

- convex if
  \[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(t) \]
  for all $x, y \in C$, $t \in [0, 1]$.
- positively homogeneous if ($C$ is a cone and)
  \[ f(tx) = tf(x) \]
  for all $x \in C$ and $t \in [0, \infty)$
- sub-additive if ($C$ is a cone and)
  \[ f(x + y) \leq f(x) + f(y) \]
  for all $x, y \in E$. 

Y. Azouzi (IPEST)
Two Theorems

We define the *lower-level set* is meant any subset of $C$ of the form

$$L(f, a) = \{ x \in C : f(x) \leq a \}, \quad \in E$$  \hspace{1cm} (1)

**Theorem**

Let $C$ be a cone in $E$. A positively homogeneous function $f : C \rightarrow E$ is convex if and only if $f$ is sub-additive.

**Theorem**

Let $C$ be a cone in the Euclidean Riesz space $\mathbb{R}^n$ and $f : C \rightarrow \mathbb{R}^+$ be a positively homogeneous function. If $L(f, 1)$ is convex then $f$ is a convex function.
Generalization

Theorem
Let $C$ be a cone in $E$. A positively homogeneous function $f : C \rightarrow E^+$ is convex if and only if $f$ is sub-additive.

Theorem
Let $C$ be a cone $C$ in $E$ and $f : C \rightarrow E^+$ be a positively homogeneous function. Then $f$ is convex if and only if the $L(f, e)$ is a convex set.

- Is it true?
Generalization

**Theorem**
Let $C$ be a cone in $E$. A positively homogeneous function $f : C \to E_+$ is convex if and only if $f$ is sub-additive.

**Theorem**
Let $C$ be a cone $C$ in $E$ and $f : C \to E^+$ be a positively homogeneous function. Then $f$ is convex if and only if the $L(f, e)$ is a convex set.

- Is it true?
- YES
Generalization

Theorem
Let $C$ be a cone in $E$. A positively homogeneous function $f : C \to E_+$ is convex if and only if $f$ is sub-additive.

Theorem
Let $C$ be a cone $C$ in $E$ and $f : C \to E^+$ be a positively homogeneous function. Then $f$ is convex if and only if the $L(f, e)$ is a convex set.

- Is it true?
- YES
- NO
What 's the "good" statement?

- Let $C$ be a cone in $E$. 

Proof.

It is enough to show that $f$ is sub-additive. Let $x, y \in C$ and put $z = \|x\| + \|y\| + \|f(x)\| + \|f(y)\| + \|f(x+y)\|$. The ideal $E_z$ is an AM-space with $z$ as a strong order unit.

By Kakutani Representation Theorem we may assume that $E_z = C(K)$, where $K$ is compact and Hausdorff and $z = 1_K$. ...
What ’s the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

An increasing positively homogeneous function $f : C \to E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.
What's the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

An increasing positively homogeneous function $f : C \to E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.

**Proof.**

It is enough to show that $f$ is sub-additive. Let $x, y \in C$ and put $z = \|x\| + \|y\| + \|f(x)\| + \|f(y)\| + \|f(x + y)\|$.
What's the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

An increasing positively homogeneous function $f : C \rightarrow E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.

**Proof.**

- It is enough to show that $f$ is sub-additive.
What's the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

An increasing positively homogeneous function $f : C \to E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.

**Proof.**

- It is enough to show that $f$ is sub-additive.
- Let $x, y \in C$ and put $z = |x| + |y| + |f(x)| + |f(y)| + |f(x+y)|$. 

The ideal $E_z$ is an AM-space with $z$ as a strong order unit. By Kakutani Representation Theorem we may assume that $E_z = C(K)$, where $K$ is compact and Hausdorff and $z = 1_K$. 

...
What's the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

*An increasing positively homogeneous function $f : C \to E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.*

**Proof.**

- It is enough to show that $f$ is sub-additive.
- Let $x, y \in C$ and put $z = |x| + |y| + |f(x)| + |f(y)| + |f(x + y)|$.
- The ideal $E_z$ is an $AM$-space with $z$ as a strong order unit.
What's the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

An increasing positively homogeneous function $f : C \to E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.

**Proof.**

- It is enough to show that $f$ is sub-additive.
- Let $x, y \in C$ and put $z = |x| + |y| + |f(x)| + |f(y)| + |f(x + y)|$.
- The ideal $E_z$ is an AM-space with $z$ as a strong order unit.
- By Kakutani Representation Theorem we may assume that $E_z = C(K)$, where $K$ is compact and Hausdorff and $z = 1_K$. 
What's the "good" statement?

- Let $C$ be a cone in $E$.

**Theorem**

An increasing positively homogeneous function $f : C \rightarrow E^+$ is convex if and only if $L(f, a)$ is convex for all $a \in E$.

**Proof.**

- It is enough to show that $f$ is sub-additive.
- Let $x, y \in C$ and put
  $$z = |x| + |y| + |f(x)| + |f(y)| + |f(x + y)|.$$ 
- The ideal $E_z$ is an AM-space with $z$ as a strong order unit.
- By Kakutani Representation Theorem we may assume that $E_z = C(K)$, where $K$ is compact and Hausdorff and $z = 1_K$.
- ....
Convex function still convex

**Theorem**

Let $f \in \mathbb{R}^\mathbb{R}$ be a convex function and $C$ be a sublattice cone in $E^+$ which contains $e$. If $f(x)$ exists in $E$ for all $x \in C$ then $f$ is convex on $C$. 
T is an averaging operator

- Let $T$ be a conditional expectation.
T is an averaging operator

- Let $T$ be a conditional expectation.
- We denote by $L^1(T)$ the natural domain of $T$. 
T is an averaging operator

- Let $T$ be a conditional expectation.
- We denote by $L^1(T)$ the natural domain of $T$.

**Theorem (Kuo-Labushagne-Watson, 2005)**

Let $E$ be a Dedekind complete $f$-algebra with order unit $e$ and $T$ be a conditional expectation operator $T$ on $E$ with $Te = e$. Then $T$ is an averaging operator, i.e., $T(xy) = xTy$, for $y \in E$, $x \in R(T)$. 
T is an averaging operator

- Let \( T \) be a conditional expectation.
- We denote by \( L^1(T) \) the natural domain of \( T \).

**Theorem (Kuo-Labushagne-Watson, 2005)**

Let \( E \) be a Dedekind complete \( f \)-algebra with order unit \( e \) and \( T \) be a conditional expectation operator \( T \) on \( E \) with \( Te = e \). Then \( T \) is an averaging operator, i.e., \( T(xy) = xTy \), for \( y \in E, \ x \in R(T) \).

**Theorem (K-L-W, 2005)**

Let \( E \) be a Dedekind complete Riesz space with weak order unit and \( T \) a conditional expectation on \( E \). Then extension \( T : L^1(T) \to L^1(T) \) is an averaging operator, i.e.,

\[
T(xy) = xT(y) \quad \text{for all} \ x \in R(T) \ \text{and} \ y \in L^1(T) \ \text{with} \ xy \in L^1(T).
\]
The range of $T$

1. By definition, the range of $T$ is a Dedekind complete Riesz subspace of $E$. 
The range of $T$

1. By definition, the range of $T$ is a Dedekind complete Riesz subspace of $E$.

**Theorem**

Let $T$ be a positive conditional expectation with domain $L^1(T)$.
The range of $T$

1. By definition, the range of $T$ is a Dedekind complete Riesz subspace of $E$.

**Theorem**

Let $T$ be a positive conditional expectation with domain $L^1(T)$.

1. The range $R(T)$ of $T$ is an $f$-subalgebra of $L^1(T)^u$. 
The range of $T$

1. By definition, the range of $T$ is a Dedekind complete Riesz subspace of $E$.

**Theorem**

Let $T$ be a positive conditional expectation with domain $L^1(T)$.

1. The range $R(T)$ of $T$ is an $f$-subalgebra of $L^1(T)^u$.
2. $R(T)L^1(T) \subseteq L^1(T)$. 

Y. Azouzi (IPEST)
Lp(T)-Spaces for finite p

Let \( p \in [1, \infty) \) and let \( T \) be a CE.
Lp(T)-Spaces for finite p

Let $p \in [1, \infty)$ and let $T$ be a CE.

Define

$$L^p(T) = \{ x \in L^1(T) : |x|^p \in L^1(T) \}.$$ 

and

$$N_p(x) = T(|x|^p)^{1/p} \text{ for all } x \in L^p(T)$$.
Lp(T)-Spaces for finite p

- Let $p \in [1, \infty)$ and let $T$ be a CE.
- Define

$$L^p(T) = \left\{ x \in L^1(T) : |x|^p \in L^1(T) \right\}.$$

and

$$N_p(x) = T (|x|^p)^{1/p} \text{ for all } x \in L^p(T).$$

**Theorem**

*Under these assumptions $L^p(T)$ is an order ideal in $L^1(T)$.***
Lp(T)-Spaces for finite p

- Let $p \in [1, \infty)$ and let $T$ be a CE.
- Define

$$L^p(T) = \{ x \in L^1(T) : |x|^p \in L^1(T) \} .$$

and

$$N_p(x) = T(|x|^p)^{1/p} \text{ for all } x \in L^p(T) .$$

**Theorem**

*Under these assumptions $L^p(T)$ is an order ideal in $L^1(T)$.***

**Theorem**

*Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$. Then*

$$N_p(x + y) \leq N_p(x) + N_p(y) \text{ for all } x, y \in L^p(T) .$$
Inequalities

Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. 

Young Inequality

$|xy| \leq \left(\frac{1}{p} |x|^{p} + \frac{1}{q} |y|^{q}\right)$ for all $x \in L^p(T)$ and $y \in L^q(T)$. 

Hölder Inequality

If $x \in L^p(T)$ and $y \in L^q(T)$ then $xy \in L^1(T)$ and $\|xy\|_1 \leq \|x\|_p \|y\|_q$. 

Lyapunov Inequality

$\|L^p(T)\|_q \|L^q(T)\|_p$ for all $x \in L^p(T)$. 

Y. Azouzi (IPEST)
Inequalities

- Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

- **Young Inequality**

  $$|xy| \leq \frac{1}{p} |x|^p + \frac{1}{q} |y|^q$$  for all $x \in L^p(T)$ and $y \in L^q(T)$.
Inequalities

- Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

- **Young Inequality**
  \[ |xy| \leq \frac{1}{p} |x|^p + \frac{1}{q} |y|^q \text{ for all } x \in L^p(T) \text{ and } y \in L^q(T). \]

- **Hölder Inequality**
  If $x \in L^p(T)$ and $y \in L^q(T)$ then
  \[ xy \in L^1(T) \text{ and } N_1(xy) \leq N_p(x) N_q(y). \]
Inequalities

- Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

- **Young Inequality**

  $$|xy| \leq \frac{1}{p} |x|^p + \frac{1}{q} |y|^q \text{ for all } x \in L^p(T) \text{ and } y \in L^q(T).$$

- **Hölder Inequality**

  If $x \in L^p(T)$ and $y \in L^q(T)$ then

  $$xy \in L^1(T) \text{ and } N_1(xy) \leq N_p(x) N_q(y).$$

- **Lyapunov Inequality**

  $$L^p(T) \subset L^q(T) \text{ and } N_q(x) \leq N_p(x) \text{ for all } x \in L^p(T).$$
More inequalities

Again $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$.

- **Chebychev Inequality**
More inequalities

Again $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$.

- **Chebychev Inequality**

**Theorem**

Let $0 \leq x \in L^p(T)$ and $u \in R(T)$. Then the following holds

$$u^p TP(x-u) + e \leq Tx^p.$$
Again $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$.

- **Chebychev Inequality**

**Theorem**

Let $0 \leq x \in L^p(T)$ and $u \in R(T)$. Then the following holds

$$u^p TP(x-u) + e \leq Tx^p.$$ 

- **Another one**
More inequalities

Again $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$.

- **Chebychev Inequality**

**Theorem**

Let $0 \leq x \in L^p(T)$ and $u \in R(T)$. Then the following holds

$$u^p TP(x-u) + e \leq Tx^p.$$ 

- **Another one**

**Theorem**

Under the same assumptions we have

$$u^{p-1} TP(x-u) + x \leq Tx^p.$$
T-uniform family

- The $T$-uniformity is an efficient tool in Martingale Theory.
T-uniform family

- The $T$-uniformity is an efficient tool in Martingale Theory.

**Definition (Kuo-Vardy-Watson, 2013)**

A family $(x_\alpha)$ in $E$ is called $T$-uniform if

$$\sup_{\alpha} TP(x_\alpha - ce)^+ |x_\alpha| \to 0 \text{ as } c \to \infty.$$
T-uniform family

- The $T$-uniformity is an efficient tool in Martingale Theory.

**Definition (Kuo-Vardy-Watson, 2013)**

A family $(x_\alpha)$ in $E$ is called $T$-uniform if

$$\sup_{\alpha} TP((x_\alpha - ce)^+) \cdot |x_\alpha| \rightarrow 0 \text{ as } c \rightarrow \infty.$$ 

**Theorem**

Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 < p < \infty$. Let $(x_\alpha)_{\alpha \in \Lambda}$ be a family in $L^1(T)$ which is bounded in $L^p(T)$, i.e., there exists $y \in L^1(T)$ such that

$$T(|x_\alpha|^p) \leq y \text{ for all } \alpha \in \Lambda.$$ 

Then $(x_\alpha)$ is $T$-uniform.
**Theorem**

Let $T$ be a conditional expectation with natural domain $L^1(T)$ and $1 \leq p < \infty$. A locally bounded net $(x_\alpha)$ in $L^1(T)$ converges to $x$ in $L^p(T)$ if and only if $(|x_\alpha|^p)$ has a $T$-uniform tail and converges to $x$ in $T$-conditionally probability.
The space \( L_\infty \).

- If \((\Omega, \mathcal{F}, \mu)\) is a probability space, then \( L^\infty (\mu) \) is given by

\[
L^\infty (\mu) = \{ f \in L^1 (\mu) : |f| \leq \lambda \text{ for some } \lambda \in \mathbb{R} \}.
\]
The space $L_\infty$.

- If $(\Omega, \mathcal{F}, \mu)$ is a probability space, then $L^\infty(\mu)$ is given by
  \[ L^\infty(\mu) = \{ f \in L^1(\mu) : |f| \leq \lambda \text{ for some } \lambda \in \mathbb{R} \} . \]

- This is a Banach space with respect to the infinity norm defined by
  \[ \|f\|_\infty = \inf \{ \lambda > 0 : |f| \leq \lambda \} \text{ for all } f \in L^\infty(\mu) . \]
The space $L_\infty$.

- If $(\Omega, \mathcal{F}, \mu)$ is a probability space, then $L^\infty (\mu)$ is given by
  
  $$ L^\infty (\mu) = \{ f \in L^1 (\mu) : |f| \leq \lambda \text{ for some } \lambda \in \mathbb{R} \} . $$

- This is a Banach space with respect to the infinity norm defined by
  
  $$ \| f \|_\infty = \inf \{ \lambda > 0 : |f| \leq \lambda \} \text{ for all } f \in L^\infty (\mu) . $$

- One of the classical results stipulates that
  
  $$ L^\infty (\mu) = \left\{ f \in \bigcap_{1 \leq p < \infty} L^p (\mu) : \lim_{p \to \infty} \| f \|_p < \infty \right\} \quad (2) $$

  and

  $$ \| f \|_\infty = \lim_{p \to \infty} \| f \|_p \text{ for all } f \in L^\infty (\mu) . \quad (3) $$
The space $L_\infty$.

- What's $L_\infty(T)$?
The space $L_\infty$.

- What’s $L^\infty(T)$?

**Definition (L-W, 2010)**

$$L^\infty(T) = \{ x \in L^1(T) : |x| \leq \lambda e \text{ for some } \lambda \in \mathbb{R} \}.$$
The space $L_\infty$.

- What's $L^\infty(T)$?

**Definition (L-W, 2010)**

$$L^\infty(T) = \{ x \in L^1(T) : |x| \leq \lambda e \text{ for some } \lambda \in \mathbb{R} \}.$$ 

- The vector-valued norm $N_\infty$ should be defined:

$$N_\infty(x) = \inf \{ \lambda e : |x| \leq \lambda e \}.$$
The space $L_\infty$.

- What's $L^\infty (T)$?

**Definition (L-W, 2010)**

$$L^\infty (T) = \{ x \in L^1 (T) : |x| \leq \lambda e \text{ for some } \lambda \in \mathbb{R} \} .$$

- The vector-valued norm $N_\infty$ should be defined:

$$N_\infty (x) = \inf \{ \lambda e : |x| \leq \lambda e \} .$$

- But,... there is a problem !!
The "right" definition

- Does it exist?
The "right" definition

- Does it exist?

  Yes,

  \[
  L^\infty (T) = \left\{ x \in L^1 (T) : |f| \leq u \text{ for some } u \in R (T) \right\}.
  \]

  and

  \[
  N_\infty (x) = \inf \left\{ u \in R (T) : |x| \leq u \right\}, \quad x \in L^\infty (T).
  \]
Properties of $L_\infty(T)$

Let $T$ be a conditional expectation with natural domain $L^1(T)$

**Theorem**

The following hold

1. $L^\infty(T)$ is an $f$-subalgebra of $L^1(T)^u$.
2. $L^\infty(T) L^p(T) \subset L^p(T)$ for $p \in [1, \infty]$.

**Theorem**

Let $x \in L^1(T)$.

1. The following are equivalent
   1. $x \in L^\infty(T)$;
   2. $x \in \bigcap_{1 \leq p < \infty} L^p(T)$ and \{ $N_p(x)$ \}$_{p \in [1, \infty)}$ is bounded in $L^1(T)$.

2. In this case we have the following formula

   $$N_\infty(x) = \sup \{ N_p(x) : p \in [1, \infty) \}.$$
Thank you