SPECTRUM OF A WEAKLY HYPERCYCLIC OPERATOR MEETS THE UNIT CIRCLE

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Abstract. It is shown that every component of the spectrum of a weakly hypercyclic operator meets the unit circle. The proof is based on the lemma that a sequence of vectors in a Banach space whose norms grow at geometrical rate doesn't have zero in its weak closure.

Suppose that T is a bounded operator on a nonzero Banach space X. Given a vector $x \in X$, we say that x is ***hypercyclic*** for T if the orbit $\mathrm{Orb}_T x = \{T^n x\}_n$ is dense in X. Similarly, x is said to be ***weakly hypercyclic*** if $\mathrm{Orb}_T x$ is weakly dense in X. A bounded operator is called ***hypercyclic*** or ***weakly hypercyclic*** if it has a hypercyclic or, respectively, a weakly hypercyclic vector. It is shown in [CS] that a weakly hypercyclic vector need not be hypercyclic, and there exist weakly hypercyclic operators which are not hypercyclic. C. Kitai showed in [K] that every component of the spectrum of a hypercyclic operator intersects the unit circle. K. Chan and R. Sanders asked in [CS] if the spectrum of a weakly hypercyclic operator meets the unit circle. In this note we show that every component of the spectrum of a weakly hypercyclic operator meets the unit circle.

Lemma 1. *Let X be a Banach space and let $c > 1$. Suppose that $x_n \in X$ satisfies $\|x_n\| \geqslant c^n$ for all $n \geqslant 1$. Then $0 \notin \overline{\{x_n\}_n}^w$.*

Proof. Let N be the smallest positive integer such that $c^N > 2$. We shall prove that there exist $F_1, \dots, F_N \in X^*$ such that

$$\max_{1 \leqslant k \leqslant N} \bigl|F_k(x_n)\bigr| \geqslant 1 \qquad (n \geqslant 1). \tag{1}$$

Since $\|x_n\| \geqslant c^n$, by replacing x_n by $(c^n/\|x_n\|)x_n$, it suffices to prove (1) for the case in which $\|x_n\| = c^n$ for all $n \geqslant 1$. First suppose that $c > 2$, so that $N = 1$. We have to construct $F_1 \in X^*$ such that $\bigl|F_1(x_n)\bigr| \geqslant 1$ for all $n \geqslant 1$. First choose $f_1 \in X^*$ with $f_1(x_1) = 1$. Then either $\bigl|f_1(x_2)\bigr| < 1$ or $\bigl|f_1(x_2)\bigr| \geqslant 1$. In the former case the Hahn-Banach theorem guarantees the existence of $g_2 \in X^*$ such that $\|g_2\| \leqslant 1/\|x_2\| = c^{-2}$, $\bigl|g_2(x_2)\bigr| = 1 - \bigl|f_1(x_2)\bigr|$, and $\bigl|(f_1 + g_2)(x_2)\bigr| = 1$. In the latter case, set $g_2 = 0$. Note that

$$\bigl|(f_1 + g_2)(x_1)\bigr| \geqslant 1 - \|g_2\|\|x_1\| \geqslant 1 - c^{-1}.$$

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Set $f_2 = f_1 + g_2$. Repeating this argument, we can find $g_3 \in X^*$ such that $\|g_3\| \leqslant 1/\|x_3\| = c^{-3}$ and $\bigl|(f_2 + g_3)(x_3)\bigr| \geqslant 1$. Note that

$$\bigl|(f_2 + g_3)(x_1)\bigr| \geqslant \bigl|f_2(x_1)\bigr| - \|g_3\|\|x_1\| \geqslant 1 - c^{-1} - c^{-2}$$

and also that

$$\bigl|(f_2 + g_3)(x_2)\bigr| \geqslant 1 - \|g_3\|\|x_2\| \geqslant 1 - c^{-1}.$$

Set $f_3 = f_2 + g_3$. Continuing in this way we obtain $f_n \in X^*$ such that (setting $g_n = f_n - f_{n-1}$) $\|g_n\| \leqslant c^{-n}$ and

$$\bigl|f_n(x_k)\bigr| \geqslant 1 - \sum_{i=1}^{n-k} c^{-i} \qquad (1 \leqslant k \leqslant n). \tag{2}$$

Thus, $\{f_n\}_n$ is norm-convergent in X^* to some $f \in X^*$. From (2), we obtain (since $c > 2$)

$$\bigl|f(x_k)\bigr| = \lim_n \bigl|f_n(x_k)\bigr| \geqslant 1 - \sum_{i=1}^{\infty} c^{-i} = \frac{c-2}{c-1} > 0.$$

Set $F_1 = (c-1)(c-2)^{-1} f$, to complete the proof in the case $c > 2$.

Now suppose that $1 < c < 2$. Set $\alpha = c^N > 2$. For each $1 \leqslant k \leqslant N$, consider the sequence $y_n = x_{k+(n-1)N}$ $(n \geqslant 1)$. Then $\|y_n\| = (c^k/\alpha)\alpha^n$. Since $\alpha > 2$ there exists $F_k \in X^*$ such that $\bigl|F_k(y_n)\bigr| \geqslant 1$ for all $n \geqslant 1$, which proves (1). ☐

Remark 2. Recall that a closed subspace Y of X^* is said to be ***norming*** if there exists $C > 0$ such that

$$\|x\| \leqslant C \sup\bigl\{\bigl|f(x)\bigr| : f \in Y,\ \|f\| \leqslant 1\bigr\} \qquad (x \in X).$$

The argument of Lemma 1 easily generalizes to give the following result. Suppose that Y is norming for X and that $\{x_n\}_n$ is a sequence in X satisfying $\|x_n\| \geqslant c^n$, where $c > 1$. Then 0 does not belong to the $\sigma(X, Y)$-closure of $\{x_n\}_n$. In particular, Lemma 1 is valid for the weak-star topology when X is a dual space.

We also make use of the following simple numerical fact. If (t_n) is a sequence in $\mathbb{R}^+ \cup \{\infty\}$, then

$$\limsup_{n\to\infty} \sqrt[n]{t_n} = \inf\{\,\nu > 0 \mid \lim_{n\to\infty} \tfrac{t_n}{\nu^n} = 0\,\} = \inf\{\,\nu > 0 \mid \limsup_{n\to\infty} \tfrac{t_n}{\nu^n} < \infty\,\}.$$

In particular, if T is a bounded operator with spectral radius r, then the Gelfand formula $\lim_n \sqrt[n]{\|T^n\|} = r$ yields that $\frac{\|T^n\|}{\lambda^n} \to 0$ for every scalar λ with $|\lambda| > r$.

Theorem 3. *If T is weakly hypercyclic, then every component of $\sigma(T)$ meets $\{z : |z| = 1\}$.*

Proof. Let x be a weakly hypercyclic vector for T. Let σ be a non-empty component of $\sigma(T)$, denote $\sigma' = \sigma(T) \setminus \sigma$. Denote by X_σ and $X_{\sigma'}$ the corresponding spectral subspaces, then X_σ and $X_{\sigma'}$ are closed, T-invariant, and $X = X_\sigma \oplus X_{\sigma'}$. Also, $\sigma(T_{|X_\sigma}) = \sigma$ and $\sigma(T_{|X_{\sigma'}}) = \sigma'$. Note that σ' might be empty, in which case we have $X_\sigma = X$ and $X_{\sigma'} = \{0\}$.

Denote by P_σ the spectral projection corresponding to σ, then $X_\sigma = \mathrm{Range} P_\sigma$. Denote $y = P_\sigma x$. Without loss of generality, $\|y\| = 1$. Since P_σ is bounded and, therefore, weakly continuous, and $\mathrm{Orb}_T y = P_\sigma\bigl(\mathrm{Orb}_T x\bigr)$, we conclude that $\mathrm{Orb}_T y$ is weakly dense in X_σ. Thus, y is weakly hypercyclic for $T_{|X_\sigma}$.

Observe that the inclusion $\sigma \subseteq \{z : |z| < 1\}$ is impossible. Indeed, in this case the spectral radius of $T_{|X_\sigma}$ would be less than 1, so that $T^n y \to 0$, which contradicts y being weakly hypercyclic for $T_{|X_\sigma}$.

Finally, we show that the inclusion $\sigma \subseteq \{z : |z| > 1\}$ is equally impossible. In this case $0 \notin \sigma = \sigma(T_{|X_\sigma})$, so that $T_{|X_\sigma}$ is invertible. Denote the inverse by S. Then S is a bounded operator on X_σ and by the Spectral Mapping Theorem

$$\sigma(S) = \big\{\lambda \mid \lambda^{-1} \in \sigma(T_{|X_\sigma})\big\} \subset \big\{z : |z| < 1\big\}.$$

Therefore, $r(S) < a$ for some $0 < a < 1$. This yields $\lim_n \frac{\|S^n\|}{a^n} = 0$, so that $\|S^n\| \leqslant a^n$ for all sufficiently large n. In particular,

$$1 = \|y\| = \|S^n T^n y\| \leqslant a^n \|T^n y\|,$$

so that $\|T^n y\| \geqslant \frac{1}{a^n}$. Lemma 1 asserts that $0 \notin \overline{\{T^n y\}_n}^w$, which contradicts y being weakly hypercyclic for $T_{|X_\sigma}$. □

Proposition 4. *Suppose that Y is norming for X. If T has a hypercyclic vector for the $\sigma(X,Y)$ topology, then the spectrum of T intersects $\{z : |z| = 1\}$.*

Proof. Suppose, to derive a contradiction, that $\sigma(T)$ does not intersect the unit circle. We use the notation introduced above with $\sigma = \sigma(T) \cap \{z : |z| < 1\}$ and $\sigma' = \sigma(T) \setminus \sigma$. Let x be a hypercyclic vector for the $\sigma(X,Y)$-topology. Then $x = y + z$, where $y \in X_\sigma$ and $z \in X_{\sigma'}$. Since $\|T^n y\| \to 0$, it follows easily that z is hypercyclic, and hence that $z \neq 0$. But then there exists $c > 1$ such that $\|T^n z\| \geqslant c^n$ for all sufficiently large n, which contradicts Remark 2. □

References

[CS] Kit C. Chan and Rebecca Sanders. A weakly hypercyclic operator that is not norm hypercyclic. To appear in J. Operator Theory.

[K] Carol Kitai. *Invariant Closed Sets for Linear Operators.* PhD thesis, Univ. of Toronto, 1982.

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