UNBOUNDED NORM TOPOLOGY IN BANACH LATTICES

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ABSTRACT. A net (x_α) in a Banach lattice X is said to un-converge to a vector x if $\big\||x_\alpha - x| \wedge u\big\| \to 0$ for every $u \in X_+$. In this paper, we investigate un-topology, i.e., the topology that corresponds to un-convergence. We show that un-topology agrees with the norm topology iff X has a strong unit. Un-topology is metrizable iff X has a quasi-interior point. Suppose that X is order continuous, then un-topology is locally convex iff X is atomic. An order continuous Banach lattice X is a KB-space iff its closed unit ball B_X is un-complete. For a Banach lattice X, B_X is un-compact iff X is an atomic KB-space. We also study un-compact operators and the relationship between un-convergence and weak*-convergence.

1. Introduction and preliminaries

For a net (x_α) in a vector lattice X, we write $x_\alpha \xrightarrow{\mathrm{o}} x$ if (x_α) ***converges*** to x ***in order***. That is, there is a net (u_γ), possibly over a different index set, such that $u_\gamma \downarrow 0$ and for every γ there exists α_0 such that $|x_\alpha - x| \leqslant u_\gamma$ whenever $\alpha \geqslant \alpha_0$. We write $x_\alpha \xrightarrow{\mathrm{uo}} x$ and say that (x_α) ***uo-converges*** to x if $|x_\alpha - x| \wedge u \xrightarrow{\mathrm{o}} 0$ for every $u \in X_+$; "uo" stands for "unbounded order". For a net (x_α) in a normed lattice X, we write $x_\alpha \xrightarrow{\|\cdot\|} x$ if (x_α) converges to x in norm. We write $x_\alpha \xrightarrow{\mathrm{un}} x$

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and say that (x_α) ***un-converges*** to x if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in X_+$; "un" stands for "unbounded norm".

A variant of uo-convergence was originally introduced in [Nak48], while the term "uo-convergence" was first coined in [DeM64]. Relationships between uo, weak, and weak* convergences were investigated in [Wic77, GX14, Gao14]. Relationships between uo-convergence and almost everywhere convergence were investigated and applied in [GX14, EM16, GTX]. We refer the reader to [GTX] for a further review of properties of uo-convergence. Un-convergence was introduced in [Tro04] and further investigated in [DOT]. For unexplained terminology on vector and Banach lattices we refer the reader to [AA02, AB06]. All vector lattices are assumed to be Archimedean.

Let us start by briefly going over some of the known properties of these modes of convergence; we refer the reader to [GTX, DOT] for details. Both uo-convergence and un-convergence respect linear and lattice operations; limits are unique. In particular, $x_\alpha \xrightarrow{\text{uo}} x$ iff $|x_\alpha - x| \xrightarrow{\text{uo}} 0$; similarly, $x_\alpha \xrightarrow{\text{un}} x$ iff $|x_\alpha - x| \xrightarrow{\text{un}} 0$. For order bounded nets, uo-convergence agrees with order convergence while un-convergence agrees with norm convergence. It follows that order intervals are uo- and un-closed. For sequences in $L_p(\mu)$, where $1 \leqslant p < \infty$ and μ is a finite measure, it is easy to see that uo-convergence agrees with convergence almost everywhere, see, e.g., [DeM64, Example 2]. Under the same assumptions, un-convergence agrees with convergence in measure, see [Tro04, Example 23]. We write L_p for $L_p[0,1]$.

Suppose that X is a vector lattice. By [GTX, Corollary 3.6], every disjoint sequence in X is uo-null. Recall that a sublattice Y of X is ***regular*** if the inclusion map preserves suprema and infima of arbitrary subsets. It was shown in [GTX, Theorem 3.2] that uo-convergence is stable under passing to and from regular sublattices. That is, if (y_α) is a net in a regular sublattice Y of X then $y_\alpha \xrightarrow{\text{uo}} 0$ in Y iff $y_\alpha \xrightarrow{\text{uo}} 0$ in X (in fact, this property characterizes regular sublattices).

It is clear that if X is an order continuous normed lattice then uo-convergence implies un-convergence. Let X be a Banach lattice and (x_n) a un-null sequence in X. Then (x_n) has a uo-null subsequence by

Proposition 4.1 of [DOT]. A disjoint sequence need not be un-null. For example, the standard unit sequence (e_n) in ℓ_∞ is not un-null. However, a un-null sequence has an asymptotically disjoint subsequence. More precisely, we have the following.

Theorem 1.1. *(*[DOT, Theorem 3.2]*) Let (x_α) be a un-null net. There is an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \xrightarrow{\|\cdot\|} 0$.*

While uo-convergence need not be given by a topology, it was observed in [DOT] that un-convergence is topological. For every $\varepsilon > 0$ and non-zero $u \in X_+$, put

$$V_{\varepsilon,u} = \bigl\{x \in X \ : \ \bigl\||x| \wedge u\bigr\| < \varepsilon\bigr\}.$$

The collection of all sets of this form is a base of zero neighborhoods for a topology, and the convergence in this topology agrees with un-convergence. We will refer to this topology as *un-topology.*

Every time a new linear topology is discovered, one is expected to ask several natural questions: is this topology metrizable? Is it locally-convex? Complete? Can one characterize (relatively) compact sets? Is this topology stronger or weaker than other known topologies? In this paper, we study these and similar questions for un-topology. In other words, our motivation for this paper is to investigate topological properties of un-topology.

Throughout this paper, X will be assumed to be a Banach lattice, unless specified otherwise. We write B_X for the closed unit ball of X. It was observed in [DOT] that $x_\alpha \xrightarrow{\text{un}} x$ implies $\|x\| \leqslant \liminf\|x_\alpha\|$. This yields that B_X is un-closed.

The following facts will be used throughout the paper.

Lemma 1.2. (i) *If (x_α) is an increasing net in a vector lattice X and $x_\alpha \xrightarrow{\text{uo}} x$ then $x_\alpha \uparrow x$;*
(ii) *If (x_α) is an increasing net in a normed lattice X and $x_\alpha \xrightarrow{\text{un}} x$ then $x_\alpha \uparrow x$ and $x_\alpha \xrightarrow{\|\cdot\|} x$.*

Proof. Without loss of generality, $x_\alpha \geqslant 0$ for all α; otherwise, pick any index α_0 and consider the net $(x_\alpha - x_{\alpha_0})_{\alpha \geqslant \alpha_0}$, which converges to

$x - x_{\alpha_0}$. Since lattice operations are uo- and un-continuous, we have $x \geqslant 0$.

(i) Take any $z \in X_+$. It follows from uo-continuity of lattice operations that $x_\alpha \wedge z \xrightarrow{\text{uo}} x \wedge z$. Since the net $(x_\alpha \wedge z)$ is order bounded and increasing, this yields $x_\alpha \wedge z \xrightarrow{\text{o}} x \wedge z$ and, therefore $x_\alpha \wedge z \uparrow x \wedge z$. It follows that $x_\alpha \wedge z \leqslant x$ for every α and every $z \in X_+$. Applying this with $z = x_\alpha$ we get $x_\alpha \leqslant x$. Thus, the net (x_α) is order bounded and, therefore, $x_\alpha \xrightarrow{\text{o}} x$, hence $x_\alpha \uparrow x$.

(ii) The proof is similar and uses the fact that every monotone norm convergent net converges in order to the same limit. We note that $x_\alpha \wedge z \xrightarrow{\|\cdot\|} x \wedge z$ and, therefore, $x_\alpha \wedge z \uparrow x \wedge z$ for every $z \in X_+$. It follows that the net (x_α) is order bounded, which yields $x_\alpha \xrightarrow{\|\cdot\|} x$ and, therefore, $x_\alpha \uparrow x$. □

Recall that [DOT, Question 2.14] asks whether $x_\alpha \xrightarrow{\text{un}} 0$ implies that there exists an increasing sequence of indices (α_k) such that $x_{\alpha_k} \xrightarrow{\text{un}} 0$. The following counterexample was kindly provided to us by E. Emelyanov.

Example 1.3. Let Ω be an uncountable set; let X be the closed sublattice of $\ell_\infty(\Omega)$ consisting of all the functions with countable support. For $\omega \in \Omega$, we write e_ω for the characteristic function of $\{\omega\}$.

Let Λ be the set of all countable subsets of Ω, ordered by inclusion. For each $\alpha \in \Lambda$, pick any $\omega \notin \alpha$ and put $x_\alpha = e_\omega$. We claim that $x_\alpha \xrightarrow{\text{un}} 0$. Indeed, let $u \in X_+$; let α_0 be the support of u. Then $x_\alpha \wedge u = 0$ whenever $\alpha \geqslant \alpha_0$.

On the other hand, let (ω_k) be any sequence in Ω; we claim that the sequence (e_{ω_k}) is not un-null. Indeed, put $\beta = \{\omega_k : k \in \mathbb{N}\}$ and let u be the characteristic function of β. Then $e_{\omega_k} \wedge u = e_{\omega_k}$ for every k; hence it does not converge in norm to zero.

In particular, if (α_k) is an increasing sequence of indices in Λ then (x_{α_k}) is not un-null.

Let $e \in X_+$. Recall that the band B_e generated by e is norm closed and contains the principal ideal I_e; hence $I_e \subseteq \overline{I_e} \subseteq B_e$. Recall also that

- e is a ***strong unit*** when $I_e = X$; equivalently, for every $x \geqslant 0$ there exists $n \in \mathbb{N}$ such that $x \leqslant ne$;
- e is a ***quasi-interior point*** if $\overline{I_e} = X$; equivalently, $x \wedge ne \xrightarrow{\|\cdot\|} x$ for every $x \in X_+$;
- e is a ***weak unit*** if $B_e = X$; equivalently, $x \wedge ne \uparrow x$ for every $x \in X_+$.

In particular, strong unit $\Rightarrow$ quasi-interior point $\Rightarrow$ weak unit.

2. Strong units

It is easy to see that each $V_{\varepsilon,u}$ is solid. It is also absorbing, that is, for every $x \in X$ there exists $\lambda > 0$ such that $\lambda x \in V_{\varepsilon,u}$. The following lemma is a dichotomy: it says that $V_{\varepsilon,u}$ is either "very small" or "very large".

Lemma 2.1. *Let $\varepsilon > 0$, and $0 \neq u \in X_+$. Then $V_{\varepsilon,u}$ is either contained in $[-u, u]$ or contains a non-trivial ideal.*

Proof. Suppose that $V_{\varepsilon,u}$ is not contained in $[-u, u]$. Then there exists $x \in V_{\varepsilon,u}$ such that $x \notin [-u, u]$. Replacing x with $|x|$, we may assume that $x > 0$. Let $y = (x - u)^+$; then $y > 0$. It is an easy exercise to show that $(\lambda y) \wedge u \leqslant x \wedge u$ for every $\lambda \geqslant 0$; it follows that $\lambda y \in V_{\varepsilon,u}$. Since $V_{\varepsilon,u}$ is solid, it contains the principal ideal I_y. □

Lemma 2.2. *If $V_{\varepsilon,u}$ is contained in $[-u, u]$ then u is a strong unit.*

Proof. Let $x \in X_+$. There exists $\lambda > 0$ such that $\lambda x \in V_{\varepsilon,u}$, hence $\lambda x \in [-u, u]$. It follows that u is a strong unit. □

Recall that if e is a positive vector in X then the principal ideal I_e equipped with the norm

$$\|x\|_e = \inf\{\lambda > 0 \,:\, |x| \leqslant \lambda e\}$$

is lattice isometric to $C(K)$ for some compact Hausdorff space K, with e corresponding to the constant one function $\mathbb{1}$; see, e.g., Theorems 3.4 and 3.6 in [AA02]. If e is a strong unit in X then $I_e = X$; it is easy to see that in this case $\|\cdot\|_e$ is equivalent to the original norm; it follows that X is lattice and norm isomorphic to $C(K)$.

It is easy to see that if $x_\alpha \xrightarrow{\|\cdot\|} x$ then $x_\alpha \xrightarrow{\mathrm{un}} x$, so norm topology generally is stronger than un-topology.

Theorem 2.3. *Let X be a Banach lattice. The following are equivalent.*

(i) *Un-topology agrees with norm topology;*
(ii) *X has a strong unit.*

Proof. Suppose that un-topology and norm topology agree. It follows that $V_{\varepsilon,u}$ is contained in B_X for some $\varepsilon > 0$ and $u > 0$. By Lemma 2.1, we conclude that $V_{\varepsilon,u}$ is contained in $[-u, u]$; hence u is a strong unit by Lemma 2.2.

Suppose now that X has a strong unit. Then X is lattice and norm isomorphic to $C(K)$ for some compact Hausdorff space K. Without loss of generality, $X = C(K)$. It follows from $x_\alpha \xrightarrow{\mathrm{un}} 0$ that $|x_\alpha| \wedge \mathbb{1} \xrightarrow{\|\cdot\|} 0$. Since the norm in $C(K)$ is the sup-norm, it is easy to see that $x_\alpha \xrightarrow{\|\cdot\|} 0$. □

3. Quasi-Interior points and metrizability

Given a net (x_α) in a vector lattice with a weak unit e, then $x_\alpha \xrightarrow{\mathrm{uo}} x$ iff $|x_\alpha - x| \wedge e \xrightarrow{\mathrm{o}} 0$; see, e.g., [GTX, Corollary 3.5] (this was proved in [Kap97] in the special case when the lattice is order complete). That is, it suffices to test uo-convergence on a weak unit. Lemma 2.11 in [DOT] provides a similar statement for un-convergence and quasi-interior points. We now prove that this property actually characterizes quasi-interior points.

Theorem 3.1. *Let $e \in X_+$. The following are equivalent.*

(i) *e is a quasi-interior point;*
(ii) *For every net (x_α) in X_+, if $x_\alpha \wedge e \xrightarrow{\|\cdot\|} 0$ then $x_\alpha \xrightarrow{\mathrm{un}} 0$;*
(iii) *For every sequence (x_n) in X_+, if $x_n \wedge e \xrightarrow{\|\cdot\|} 0$ then $x_n \xrightarrow{\mathrm{un}} 0$.*

Proof. The implication (i)⇒(ii) was proved in [DOT, Lemma 2.11]. (ii)⇒(iii) is trivial. This leaves (iii)⇒(i).

Suppose (iii). Fix $x \in X_+$. We need to show that $x \wedge ne \xrightarrow{\|\cdot\|} x$ or, equivalently $(x - ne)^+ \xrightarrow{\|\cdot\|} 0$ as a sequence of n. Put $u = x \vee e$. The ideal I_u is lattice isomorphic (as a vector lattice) to $C(K)$ for some compact space K, with u corresponding to $\mathbb{1}$. Since $x, e \in I_u$, we may consider x and e as elements of $C(K)$. Note that $x \vee e = \mathbb{1}$ implies that x and e never vanish simultaneously.

For each $n \in \mathbb{N}$, we define

$$F_n = \bigl\{t \in K \,:\, x(t) \geqslant ne(t)\bigr\} \text{ and } O_n = \bigl\{t \in K \,:\, x(t) > ne(t)\bigr\}.$$

Clearly, $O_n \subseteq F_n$, O_n is open, and F_n is closed.

Claim 1: $F_{n+1} \subseteq O_n$. Indeed, let $t \in F_{n+1}$. Then $x(t) \geqslant (n+1)e(t)$. If $e(t) > 0$ then $x(t) > ne(t)$, so that $t \in O_n$. If $e(t) = 0$ then $x(t) > 0$, hence $t \in O_n$.

By Urysohn's Lemma, we find $z_n \in C(K)$ such that $0 \leqslant z_n \leqslant x$, z_n agrees with x on F_{n+1} and vanishes outside of O_n. We can also view z_n as an element of X.

Claim 2: $n(z_n \wedge e) \leqslant x$. Let $t \in K$. If $t \in O_n$ then $n(z_n \wedge e)(t) \leqslant ne(t) < x(t)$. If $t \notin O_n$ then $z_n(t) = 0$, so that the inequality is satisfied trivially.

Claim 3: $\bigl(x - (n+1)e\bigr)^+ \leqslant z_n$. Again, let $t \in K$. If $t \in F_{n+1}$ then $\bigl(x - (n+1)e\bigr)^+ \leqslant x(t) = z_n(t)$. If $t \notin F_{n+1}$ then $x(t) < (n+1)e(t)$, so that $\bigl(x - (n+1)e\bigr)^+(t) = 0$ and the inequality is satisfied trivially.

Now, Claim 2 yields $0 \leqslant z_n \wedge e \leqslant \frac{1}{n}x \xrightarrow{\|\cdot\|} 0$, so that $z_n \wedge e \xrightarrow{\|\cdot\|} 0$. By assumption, this yields $z_n \xrightarrow{\mathrm{un}} 0$. Since $0 \leqslant z_n \leqslant x$ for every n, the sequence (z_n) is order bounded and, therefore, $z_n \xrightarrow{\|\cdot\|} 0$. Now Claim 3 yields $\bigl(x - (n+1)e\bigr)^+ \xrightarrow{\|\cdot\|} 0$, which concludes the proof. ☐

Theorem 3.2. *Un-topology is metrizable iff X has a quasi-interior point. If e is a quasi-interior point then $d(x, y) = \bigl\||x - y| \wedge e\bigr\|$ is a metric for un-topology.*

Proof. Suppose that $e \in X_+$ is a quasi-interior point and put $d(x, y) = \bigl\||x - y| \wedge e\bigr\|$ for $x, y \in X$. It can be easily verified that this defines a metric on X. Indeed, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for every $x, y \in X$. If $d(x, y) = 0$ then $|x - y| \wedge e = 0$, hence $|x - y| = 0$ because e is a weak unit, so that $x = y$. The triangle inequality follows from the fact that

$$|x - z| \wedge e \leqslant |x - y| \wedge e + |y - z| \wedge e.$$

Note also that $x_\alpha \xrightarrow{\text{un}} x$ iff $d(x_\alpha, x) \to 0$ for every net (x_α) in X.

Conversely, suppose that un-topology is metrizable; let d be a metric for it. For each n, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centred at zero for the metric, that is,

$$B_{\frac{1}{n}} = \bigl\{x \in X \,:\, d(x,0) \leqslant \tfrac{1}{n}\bigr\}.$$

Since $B_{\frac{1}{n}}$ is a neighborhood of zero for the un-topology, it contains V_{ε_n,u_n} for some $\varepsilon_n > 0$ and $u_n > 0$. Let $M_n = 2^n\|u_n\| + 1$; then the series $e = \sum_{n=1}^{\infty} \frac{u_n}{M_n}$ converges. Note that $M_n > 1$ and $u_n \leqslant M_n e$ for every n. We claim that e is a quasi-interior point.

It suffices that Theorem 3.1(ii) is satisfied. Suppose that $x_\alpha \wedge e \xrightarrow{\|\cdot\|} 0$ for some net (x_α) in X_+. Fix n. It follows from

$$x_\alpha \wedge u_n \leqslant (M_n x_\alpha) \wedge (M_n e) = M_n(x_\alpha \wedge e) \xrightarrow{\|\cdot\|} 0$$

that $x_\alpha \wedge u_n \xrightarrow{\|\cdot\|} 0$. Then there exists α_0 such that $\|x_\alpha \wedge u_n\| < \varepsilon_n$ whenever $\alpha \geqslant \alpha_0$. Consequently, x_α is in V_{ε_n,u_n} and, therefore, in $B_{\frac{1}{n}}$. It follows that $x_\alpha \to 0$ in the metric, hence $x_\alpha \xrightarrow{\text{un}} 0$. □

Note that a linear Hausdorff topological space is metrizable iff it is first countable, i.e., has a countable base of neighborhoods of zero, see, e.g., [KN63, pp. 49]. Therefore, Theorem 3.2 implies, in particular, that un-topology is first countable iff X has a quasi-interior point. This should be compared with Corollary 2.13 and Question 2.14 in [DOT] (we now know from Example 1.3 that Question 2.14 has a negative answer).

Proposition 3.3. *Un-topology is stronger than or equal to a metric topology iff X has a weak unit.*

Proof. Suppose that un-topology is stronger than or equal to a topology given by a metric. Construct e as in the second part of the proof of Theorem 3.2. We claim that e is a weak unit. Suppose that $x \wedge e = 0$. It follows that $x \wedge u_n = 0$ for every n and, therefore, $x \in V_{\varepsilon_n,u_n}$, hence $x \in B_{\frac{1}{n}}$. It follows that $x = 0$.

Conversely, let $e \in X_+$ be a weak unit. For $x, y \in X$, define $d(x,y) = \bigl\|\lvert x-y\rvert \wedge e\bigr\|$. As in the first part of the proof of Theorem 3.2, this is a metric and $x_\alpha \xrightarrow{\text{un}} x$ implies $d(x_\alpha, x) \to 0$. □

When is every un-null sequence norm bounded? If X has a strong unit then, by Theorem 2.3, un-topology agrees with norm topology, hence every un-null sequence is norm null and, in particular, norm bounded. This justifies the following question: *If every un-null sequence in X is norm bounded (or even norm null), does this imply that X has a strong unit?* The following example shows that, in general, the answer in negative.

Example 3.4. Let X be as in Example 1.3. Clearly, X does not have a strong unit; it does not even have a weak unit. Yet, every un-null sequence in X is norm null. Indeed, suppose that $x_n \xrightarrow{\text{un}} 0$. Let u be the characteristic function of $\bigcup_{n=1}^{\infty} \operatorname{supp} x_n$. By assumption, $|x_n| \wedge u \xrightarrow{\|\cdot\|} 0$. It follows that for every $\varepsilon \in (0,1)$ there exists n_0 such that for every $n \geqslant n_0$ we have $\bigl\||x_n| \wedge u\bigr\| < \varepsilon$. It follows that $\|x_n\| < \varepsilon$.

However, we will see that the answer is affirmative under certain additional assumptions.

Recall that every disjoint sequence is uo-null. Thus, if $\dim X = \infty$, one can take any non-zero disjoint sequence, scale it to make it norm unbounded, and thus produce a uo-null sequence which is not norm bounded. However, this trick does not work for un-topology because a disjoint sequence need not be un-null. Moreover, we have the following.

Proposition 3.5. *The following are equivalent.*

(i) *X is order continuous;*
(ii) *Every disjoint sequence in X is un-null;*
(iii) *Every disjoint net in X is un-null.*

Proof. (i)⇒(ii) because every disjoint sequence is uo-null and, therefore, un-null. To show that (ii)⇒(i), note that every order bounded disjoint sequence is norm null and apply [AB06, Theorem 4.14].

(iii)⇒(ii) is trivial. To show that (ii)⇒(iii), suppose that there exists a disjoint net (x_α) which is not un-null. Then there exist $\varepsilon > 0$ and $u \in X_+$ such that for every α there exists $\beta > \alpha$ with $\bigl\||x_\beta| \wedge u\bigr\| > \varepsilon$. Inductively, we find an increasing sequence (α_k) of indices such that $\bigl\||x_{\alpha_k}| \wedge u\bigr\| > \varepsilon$. Hence, the sequence (x_{α_k}) is disjoint but not un-null. □

Corollary 3.6. *If X is order continuous and every un-null sequence in X is norm bounded then* $\dim X < \infty$ *(and, therefore, X has a strong unit).*

Proof. Suppose $\dim X = \infty$. Then there exists a non-zero disjoint sequence in X. Scaling it if necessary, we may assume that it is not norm bounded. Yet it is un-null. A contradiction. □

Note that Example 2.7 in [DOT] is an example of a disjoint but non un-null sequence in an infinite-dimensional Banach lattice which is not order continuous and lacks a strong unit.

Proposition 3.7. *If X has a quasi-interior point and every un-null sequence is norm bounded then X has a strong unit.*

Proof. By Theorem 3.2, the un-topology on X is metrizable. Fix such a metric. As before, for each n, let $B_{\frac{1}{n}}$ be the ball of radius $\frac{1}{n}$ centred at zero for the metric. For each n, $B_{\frac{1}{n}}$ contains V_{ε_n,u_n} for some $\varepsilon_n > 0$ and $u_n > 0$. If $V_{\varepsilon_n,u_n} \subseteq [-u_n, u_n]$ for some n then u_n is a strong unit by Lemma 2.2. Otherwise, by Lemma 2.1, each V_{ε_n,u_n} contains a non-trivial ideal. Pick any x_n in this ideal with $\|x_n\| = n$. Then the sequence (x_n) is norm unbounded; yet $x_n \in B_{\frac{1}{n}}$ for every n, so that $x_n \xrightarrow{\text{un}} 0$; a contradiction. □

4. Un-convergence in a sublattice

Recall that if (y_α) is a net in a regular sublattice Y of a vector lattice X then $y_\alpha \xrightarrow{\text{uo}} 0$ in Y iff $y_\alpha \xrightarrow{\text{uo}} 0$ in X. The situation is very different for un-convergence. Let Y be a sublattice of a normed lattice X and (y_α) a net in Y. If $y_\alpha \xrightarrow{\text{un}} 0$ in X then, clearly, $y_\alpha \xrightarrow{\text{un}} 0$ in Y. However, the following examples show that the converse fails even for closed ideals or bands.

Example 4.1. The sequence of the standard unit vectors (e_n) is un-null in c_0 but not in ℓ_∞, even though c_0 is a closed ideal in ℓ_∞.

Example 4.2. Let $X = C[-1, 1]$ and Y be the set of all $f \in X$ which vanish on $[-1, 0]$. It is easy to see that Y is a band (though it is not a projection band). Let (f_n) be a sequence in Y_+ such that $\|f_n\| = 1$

and $\operatorname{supp} f_n \subseteq [\frac{1}{n+1}, \frac{1}{n}]$. Since X has a strong unit, the un-topology on X agrees with the norm topology, hence (f_n) is not un-null in X. However, it is easy to see that (f_n) is un-null in Y.

Nevertheless, there are some good news. Recall that a sublattice Y of a vector lattice X is ***majorizing*** if for every $x \in X_+$ there exists $y \in Y_+$ with $x \leqslant y$.

Theorem 4.3. *Let Y be a sublattice of a normed lattice X and (y_α) a net in Y such that $y_\alpha \xrightarrow{\mathrm{un}} 0$ in Y. Each of the following conditions implies that $y_\alpha \xrightarrow{\mathrm{un}} 0$ in X.*

(i) *Y is majorizing in X;*
(ii) *Y is norm dense in X;*
(iii) *Y is a projection band in X.*

Proof. Without loss of generality, $y_\alpha \geqslant 0$ for every α. (i) is straightforward. To prove (ii), take $u \in X_+$ and fix $\varepsilon > 0$. Find $v \in Y_+$ with $\|u - v\| < \varepsilon$. By assumption, $y_\alpha \wedge v \xrightarrow{\|\cdot\|} 0$. We can find α_0 such that $\|y_\alpha \wedge v\| < \varepsilon$ whenever $\alpha \geqslant \alpha_0$. It follows from $u \leqslant v + |u - v|$ that $y_\alpha \wedge u \leqslant y_\alpha \wedge v + |u - v|$, so that

$$\|y_\alpha \wedge u\| \leqslant \|y_\alpha \wedge v\| + \|u - v\| < 2\varepsilon.$$

It follows that $y_\alpha \wedge u \xrightarrow{\|\cdot\|} 0$. Hence $y_\alpha \xrightarrow{\mathrm{un}} 0$ in X.

To prove (iii), let $u \in X_+$. Then $u = v + w$ for some positive $v \in Y$ and $w \in Y^d$. It follows from $y_\alpha \perp w$ that $y_\alpha \wedge u = y_\alpha \wedge v \xrightarrow{\|\cdot\|} 0$. □

Recall that every (Archimedean) vector lattice X is majorizing in its ***order (or Dedekind) completion*** X^δ; see , e.g., [AB06, p. 101].

Corollary 4.4. *If X is a normed lattice and $x_\alpha \xrightarrow{\mathrm{un}} x$ in X then $x_\alpha \xrightarrow{\mathrm{un}} x$ in the order completion X^δ of X.*

Corollary 4.5. *If X is a KB-space and $x_\alpha \xrightarrow{\mathrm{un}} 0$ in X then $x_\alpha \xrightarrow{\mathrm{un}} 0$ in X^{**}.*

Proof. By [AB06, Theorem 4.60], X is a projection band in X^{**}. The conclusion now follows from Theorem 4.3(iii). □

Example 4.1 shows that the assumption that X is a KB-space cannot be removed.

Corollary 4.6. *Let Y be a sublattice of an order continuous Banach lattice X. If $y_\alpha \xrightarrow{\text{un}} 0$ in Y then $y_\alpha \xrightarrow{\text{un}} 0$ in X.*

Proof. Suppose that $y_\alpha \xrightarrow{\text{un}} 0$ in Y. By Theorem 4.3(i), $y_\alpha \xrightarrow{\text{un}} 0$ in the ideal $I(Y)$ generated by Y in X. By Theorem 4.3(ii), $y_\alpha \xrightarrow{\text{un}} 0$ in the closure $\overline{I(Y)}$ of the ideal. Since X is order continuous, $\overline{I(Y)}$ is a projection band in X. It now follows from Theorem 4.3(iii) that $y_\alpha \xrightarrow{\text{un}} 0$ in X. $\square$

Question 4.7. Let B be a band in X. Suppose that every net in B which is un-null in B is also un-null in X. Does this imply that B is a projection band?

Proposition 4.8. *Every band in a normed lattice is un-closed.*

Proof. Let B be a band and (x_α) a net in B such that $x_\alpha \xrightarrow{\text{un}} x$. Fix $z \in B^d$. Then $|x_\alpha| \wedge z = 0$ for every α. Since lattice operations are un-continuous, we have $|x| \wedge z = 0$. It follows that $x \in B^{dd} = B$. $\square$

Remark 4.9. Let B be a projection band a normed lattice X. We write P_B for the corresponding band projection. It follows easily from $0 \leqslant P_B \leqslant I$ that if $x_\alpha \xrightarrow{\text{un}} x$ in X then $P_B x_\alpha \xrightarrow{\text{un}} P_B x$ both in X and in B.

Dense band decompositions. Let X be a Banach lattice. By a ***dense band decomposition*** of X we mean a family $\mathcal{B}$ of pairwise disjoint projection bands in X such that the linear span of all of the bands in $\mathcal{B}$ is norm dense in X.

Lemma 4.10. *Let $\mathcal{B}$ be a family of pairwise disjoint projection bands in a Banach lattice X. $\mathcal{B}$ is a dense band decomposition of X iff for every $x \in X$ and every $\varepsilon > 0$ there exist $B_1, \dots, B_n$ in $\mathcal{B}$ such that $\left\| x - \sum_{i=1}^n P_{B_i} x \right\| < \varepsilon$.*

Proof. Suppose that $\mathcal{B}$ is a dense band decomposition of X. Let $x \in X$ and $\varepsilon > 0$. By assumption, we can find distinct bands $B_1, \dots, B_n$ and vectors $x_1 \in B_1, \dots, x_n \in B_n$ such that $\left\| x - \sum_{i=1}^n x_i \right\| < \varepsilon$. Put $Q = I - \sum_{i=1}^n P_{B_i}$. Then Q is also a band projection, hence it is a lattice homomorphism and $0 \leqslant Q \leqslant I$. Note also that $Qx_i = 0$ for $i = 1, \dots, n$. We have

$$\Bigl|x - \sum_{i=1}^{n} x_i\Bigr| \geqslant Q\Bigl|x - \sum_{i=1}^{n} x_i\Bigr| = \Bigl|Qx - \sum_{i=1}^{n} Qx_i\Bigr| = \Bigl|x - \sum_{i=1}^{n} P_{B_i}x\Bigr|.$$

It follows that $\bigl\|x - \sum_{i=1}^{n} P_{B_i}x\bigr\| < \varepsilon$.

The converse implication is trivial. ☐

Our definition of a disjoint band decomposition is partially motivated by following fact.

Theorem 4.11. *(*[LT79, Proposition 1.a.9]*) Every order continuous Banach lattice admits a dense band decomposition $\mathcal{B}$ such that each band in $\mathcal{B}$ has a weak unit.*

It is easy to see that if X is an order continuous Banach lattice and $\mathcal{B}$ is a pairwise disjoint collection of bands such that $x = \sup\{P_B x : B \in \mathcal{B}\}$ for every $x \in X_+$ then $\mathcal{B}$ is a dense band decomposition.

Theorem 4.12. *Suppose that $\mathcal{B}$ is a dense band decomposition of a Banach lattice X. Then $x_\alpha \xrightarrow{\mathrm{un}} x$ in X iff $P_B x_\alpha \xrightarrow{\mathrm{un}} P_B x$ in B for each $B \in \mathcal{B}$.*

Proof. Without loss of generality, $x = 0$ and $x_\alpha \geqslant 0$ for every α. The forward implication follows immediately from Remark 4.9. To prove the converse, suppose that $P_B x_\alpha \xrightarrow{\mathrm{un}} 0$ in B for each $B \in \mathcal{B}$. Let $u \in X_+$; it suffices to show that $x_\alpha \wedge u \xrightarrow{\|\cdot\|} 0$. Fix $\varepsilon > 0$. Find $B_1, \dots, B_n \in \mathcal{B}$ such that $\bigl\|u - \sum_{i=1}^{n} P_{B_i}u\bigr\| < \varepsilon$. Since $P_{B_i}x_\alpha \xrightarrow{\mathrm{un}} 0$ in B_i as $i = 1, \dots, n$, we can find α_0 such that $\bigl\|P_{B_i}x_\alpha \wedge P_{B_i}u\bigr\| < \frac{\varepsilon}{n}$ for every $\alpha \geqslant \alpha_0$ and every $i = 1, \dots, n$. It follows from $x_\alpha \wedge P_{B_i}u \in B_i$ that $x_\alpha \wedge P_{B_i}u = P_{B_i}x_\alpha \wedge P_{B_i}u$. Therefore,

$$\begin{aligned}\|x_\alpha \wedge u\| &\leqslant \Bigl\|x_\alpha \wedge \sum_{i=1}^{n} P_{B_i}u\Bigr\| + \Bigl\|u - \sum_{i=1}^{n} P_{B_i}u\Bigr\| \leqslant \Bigl\|\sum_{i=1}^{n} x_\alpha \wedge P_{B_i}u\Bigr\| + \varepsilon \\ &= \Bigl\|\sum_{i=1}^{n} P_{B_i}x_\alpha \wedge P_{B_i}u\Bigr\| + \varepsilon \leqslant n \cdot \frac{\varepsilon}{n} + \varepsilon \leqslant 2\varepsilon.\end{aligned}$$

☐

Remark 4.13. Recall that a positive non-zero vector a in a vector lattice X is an ***atom*** if the principal ideal I_a generated by a coincides with $\operatorname{span} a$. In this case, I_a is a projection band, and the corresponding band projection P_a has form $f_a \otimes a$ for some positive functional f_a, that is, $P_a x = f_a(x)a$. We say that X is ***non-atomic*** if it has no atoms. We say that X is ***atomic*** if X is the band generated by all the atoms. In the latter case, $x = \sup\{f_a(x)a \,:\, a \text{ is an atom}\}$ for every $x \in X_+$. See, e.g., [Sch74, p. 143].

It follows that if X is an order continuous atomic Banach lattice, the family $\{I_a \,:\, a \text{ is an atom}\}$ is a dense band decomposition of X. Applying Theorem 4.12, we conclude that in such spaces un-convergence is exactly the "coordinate-wise" convergence:

Corollary 4.14. *Let X be an atomic order continuous Banach lattice. Then $x_\alpha \xrightarrow{\text{un}} x$ iff $f_a(x_\alpha) \to f_a(x)$ for every atom a.*

Remark 4.15. The order continuity assumption cannot be removed. Indeed, ℓ_∞ is atomic, the sequence (e_n) converges to zero coordinate-wise, yet it is not un-null.

The following results extends [DOT, Proposition 6.2].

Proposition 4.16. *The following are equivalent:*

(i) $x_\alpha \xrightarrow{\text{w}} 0$ *implies* $x_\alpha \xrightarrow{\text{un}} 0$ *for every net* (x_α) *in* X;
(ii) $x_n \xrightarrow{\text{w}} 0$ *implies* $x_n \xrightarrow{\text{un}} 0$ *for every sequence* (x_n) *in* X;
(iii) X *is atomic and order continuous.*

Proof. (i)⇒(ii) is trivial. The implication (ii)⇒(iii) is a part of [DOT, Proposition 6.2]. The implication (iii)⇒(i) follows from Corollary 4.14. □

5. AL-representations and local convexity

In this section, we will show that un-topology on an order continuous Banach lattice X is locally convex iff X is atomic. Our main tool is the relationship between un-convergence in X and in an AL-representation of X.

It was observed in [Tro04, Example 23] that for a net (x_α) in $L_p(\mu)$ where μ is a finite measure and $1 \leqslant p < \infty$, one has $x_\alpha \xrightarrow{\text{un}} 0$ iff $x_\alpha \xrightarrow{\mu} 0$ (i.e., the net converges to zero in measure). Note that this does not extend to σ-finite measures. Indeed, let $X = L_p(\mathbb{R})$ and let x_n be the characteristic function of $[n, n+1]$. Then $x_n \xrightarrow{\text{un}} 0$ but (x_n) does not converge to zero in measure. On the other hand, let (x_α) be a net in $L_p(\mu)$ where μ is a σ-finite measure, let (Ω_n) be a countable partition of Ω into sets of finite measure; it follows from Theorem 4.12 that $x_\alpha \xrightarrow{\text{un}} 0$ iff the restriction of x_α to Ω_n converges to zero in measure for every n.

Suppose that X is an order continuous Banach lattice with a weak unit e. By [LT79, Theorem 1.b.14], X can be represented as an ideal of $L_1(\mu)$ for some probability measure μ. More precisely, there is a lattice isomorphism from X onto a norm-dense ideal of $L_1(\mu)$; with a slight abuse of notation we will view X itself as an ideal of $L_1(\mu)$. Moreover, this representation may be chosen so that e corresponds to $\mathbb{1}$, $L_\infty(\mu)$ is a norm-dense ideal in X, and both inclusions in $L_\infty(\mu) \subseteq X \subseteq L_1(\mu)$ are continuous. We call $L_1(\mu)$ an ***AL-representation*** for X and e. Let (x_n) be a sequence in X. It was shown in [GTX, Remark 4.6] that $x_n \xrightarrow{\text{uo}} 0$ in X iff $x_n \xrightarrow{\text{a.e.}} 0$ in $L_1(\mu)$. It was shown in [DOT, Theorem 4.6] that $x_n \xrightarrow{\text{un}} 0$ in X iff $x_n \xrightarrow{\mu} 0$ in $L_1(\mu)$. Since un-topology and the topology of convergence in measure are both metrizable on X because X has a weak unit, it follows that these two topologies coincide on X. In particular, $x_\alpha \xrightarrow{\text{un}} 0$ in X iff $x_\alpha \xrightarrow{\mu} 0$ in $L_1(\mu)$ for every net (x_α) in X. This may also be deduced from Amemiya's Theorem (see, e.g., Theorem 2.4.8 in [MN91]) as follows:

$$x_\alpha \xrightarrow{\text{un}} 0 \text{ in } X \quad\Leftrightarrow\quad \|x_\alpha \wedge e\|_X \to 0 \quad\overset{\text{Amemiya}}{\Leftrightarrow}\quad \|x_\alpha \wedge \mathbb{1}\|_{L_1} \to 0 \quad\Leftrightarrow\quad x_\alpha \xrightarrow{\mu} 0 \text{ in } L_1(\mu)$$

for every net (x_α) in X_+.

Proposition 5.1. *Let X be a non-atomic order continuous Banach lattice and W a neighborhood of zero for un-topology. If W is convex then $W = X$.*

Proof. Fix $e \in X_+$; we will show that $e \in W$. We know that $V_{\varepsilon,u} \subseteq W$ for some $\varepsilon > 0$ and $u > 0$. Consider the principal band B_e. Since X is order continuous, B_e is a projection band in X; let P_e be the corresponding band projection. Furthermore, B_e is a non-atomic order continuous Banach lattice with a weak unit. Let $L_1(\Omega,\mathcal{F},\mu)$ be an AL-representation for B_e with $e=\mathbb{1}$. Note that the measure μ is non-atomic because if a measurable set A were an atom for μ then its characteristic function χ_A would be an atom in X. Fix $n\in\mathbb{N}$. Using the non-atomicity of μ, we find a measurable partition $A_{n,1},\dots,A_{n,n}$ of Ω with $\mu(A_{n,i})=\frac{1}{n}$ as $i=1,\dots,n$; see, e.g., Exercise 2 in [Hal70, p. 174]. Since $L_\infty(\mu)\subseteq B_e\subseteq L_1(\mu)$, we may view the characteristic functions $\chi_{A_{n,i}}$ as elements of B_e. Consider the vectors $(n\chi_{A_{n,i}})\wedge u$ as $i=1,\dots,n$; they belong to B_e, so that we may view them as functions in $L_1(\mu)$. Let g_n be the function in this list whose norm in X is maximal; if there are more than one, pick any one. Repeating this construction for every $n\in\mathbb{N}$, we produce a sequence (g_n) in $[0,u]\cap B_e$. It follows that $g_n\leqslant P_eu$ for every n. Since P_eu may be viewed as an element of $L_1(\mu)$ and the measure of the support of g_n tends to zero, it follows that $\|g_n\|_{L_1}\to 0$. Amemiya's Theorem yields $\|g_n\|_X\to 0$. Fix n such that $\|g_n\|_X<\varepsilon$. It follows from the definition of g_n that $\big\|(n\chi_{A_{n,i}})\wedge u\big\|_X<\varepsilon$ as $i=1,\dots,n$, so that $n\chi_{A_{n,i}}$ is in $V_{\varepsilon,u}$ and, therefore, in W. Since W is convex and

$$e=\mathbb{1}=\frac{1}{n}\sum_{i=1}^{n}n\chi_{A_{n,i}},$$

we have $e\in W$. Therefore, $X_+\subseteq W$. Furthermore, it follows from $n\chi_{A_{n,i}}\in V_{\varepsilon,u}$ that $-n\chi_{A_{n,i}}\in V_{\varepsilon,u}$ for all $i=1,\dots,n$ and, therefore, $-e\in W$. This yields $X_-\subseteq W$. Finally, for every $x\in X$ we have $x=\frac{1}{2}\big(2x^++2(-x^-)\big)$, so that $x\in W$.

□

Theorem 5.2. *Let X be an order continuous Banach lattice. Un-topology on X is locally convex iff X is atomic.*

Proof. Suppose that X is atomic. By Corollary 4.14, un-topology is determined by the family of seminorms $x\mapsto\big|f_a(x)\big|$ where a is an atom of X; hence the topology is locally convex.

Suppose that un-topology is locally convex but X is not atomic. It follows that there is $e\in X_+$ such that B_e is non-atomic. By Theorem 4.3, un-topology on B_e agrees with the relative topology induced on B_e by un-topology on X; in particular, it is locally convex. On the other hand, Proposition 5.1 asserts that this topology on B_e has no proper convex neighborhoods; a contradiction. □

Un-continuous functionals. Theorem 5.2 allows us to describe un-continuous linear functionals. For a functional $\varphi \in X^*$, we say that φ is ***un-continuous*** if it is continuous with respect to the un-topology on X or, equivalently, if $x_\alpha \xrightarrow{\text{un}} 0$ implies $\varphi(x_\alpha) \to 0$.

Proposition 5.3. *The set of all un-continuous functionals in X^* is an ideal.*

Proof. It is straightforward to verify that this set is a linear subspace. Suppose that φ in X^* is un-continuous; we will show that $|\varphi|$ is also un-continuous. Fix $\delta > 0$. One can find $\varepsilon > 0$ and $u > 0$ such that $\bigl|\varphi(x)\bigr| < \delta$ whenever $x \in V_{\varepsilon,u}$. Fix $x \in V_{\varepsilon,u}$. Since $V_{\varepsilon,u}$ is solid, $|y| \leqslant |x|$ implies $y \in V_{\varepsilon,u}$ and, therefore, $\bigl|\varphi(y)\bigr| < \delta$. By the Riesz-Kantorovich formula, we get

$$\bigl||\varphi|(x)\bigr| \leqslant |\varphi|\bigl(|x|\bigr) = \sup\bigl\{\bigl|\varphi(y)\bigr| \;:\; |y| \leqslant |x|\bigr\} \leqslant \delta.$$

It follows that $|\varphi|$ is un-continuous. Hence, the set of all un-continuous functionals in X^* forms a sublattice. It is easy to see that if $\varphi \in X^*_+$ is un-continuous and $0 \leqslant \psi \leqslant \varphi$ then ψ is also un-continuous; this completes the proof. □

Recall that if a is an atom then f_a stands for the corresponding "coordinate functional".

Corollary 5.4. *Suppose that X is an order continuous Banach lattice and $\varphi \in X^*$ is un-continuous.*

(i) *If X is atomic then $\varphi = \lambda_1 f_{a_1} + \dots + \lambda_n f_{a_n}$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $a_1, \dots, a_n$ are atoms;*
(ii) *If X is non-atomic then $\varphi = 0$.*

Proof. By Proposition 5.3, we may assume that $\varphi \geqslant 0$; otherwise we consider φ^+ and φ^-.

Suppose X is atomic; let A be a maximal disjoint family of atoms. We claim that the set $F := \{a \in A : \varphi(a) \neq 0\}$ is finite. Indeed, otherwise, take a sequence (a_n) of distinct atoms in F and put $x_n = \frac{1}{\varphi(a_n)} a_n$.

Then $x_n \xrightarrow{\text{un}} 0$ by Corollary 4.14, yet $\varphi(x_n) = 1$; a contradiction. This proves the claim.

Since X is order continuous, it follows from Remark 4.13 that X has a disjoint band decomposition $X = B_F \oplus B_{A\setminus F}$. Since $\varphi(a) = 0$ for all $a \in A \setminus F$, φ vanishes on the ideal $I_{A\setminus F}$ and, therefore, on $B_{A\setminus F}$ because φ is order continuous. On the other hand, since F is finite, $B_F = \operatorname{span} F$ and, therefore, is finite-dimensional. It follows that φ is a linear combination of $\{f_a : a \in F\}$.

Suppose now that X is non-atomic. Let $W = \varphi^{-1}(-1, 1)$. Then W is a convex neighborhood of zero for the un-topology. By Proposition 5.1, $W = X$. This easily implies $\varphi = 0$. □

Case (i) of the preceding corollary essentially says that every un-continuous functional on an atomic order continuous space has finite support.

Example 5.5. Let $X = \ell_2$. By Corollary 5.4, the set of all un-continuous functionals in X^* may be identified with c_{00}, the linear subspace of all sequences with finite support. Clearly, it is neither norm closed nor order closed; it is not even σ-order closed in X^*.

Example 5.6. Let $X = C_0(\Omega)$ where Ω is a locally compact Hausdorff topological space. It was observed in [Tro04, Example 20] that the un-topology in X agrees with the topology of uniform convergence on compact subsets of Ω.

Let $\varphi \in X^*_+$. By the Riesz Representation Theorem, there exists a regular Borel measure μ such that $\varphi(f) = \int f\,d\mu$ for every $f \in X$; see, e.g., [Con99, Theorem III.5.7]. An argument similar to the proof of [Con99, Proposition IV.4.1] shows that φ is un-continuous iff μ has compact support.

6. Un-completeness

Throughout this section, X is assumed to be an order continuous Banach lattice. Since un-topology is linear, one can talk about un-Cauchy nets. That is, a net (x_α) is un-Cauchy if for every un-neighborhood U of zero there exists α_0 such that $x_\alpha - x_\beta \in U$ whenever $\alpha, \beta \geqslant \alpha_0$. We investigate whether X itself or some "nice" subset of X is un-complete. First, we observe that the entire space is un-complete only when X is finite-dimensional.

Lemma 6.1. *Let (x_n) be a positive disjoint sequence in an order continuous Banach lattice X such that (x_n) is not norm null. Put $s_n = \sum_{i=1}^n x_i$. Then (s_n) is un-Cauchy but not un-convergent.*

Proof. The sequence (s_n) is monotone increasing and does not converge in norm; hence it is not un-convergent by Lemma 1.2(ii). To show that (s_n) is un-Cauchy, fix any $\varepsilon > 0$ and a non-zero $u \in X_+$. Since x_i's are disjoint, we have $s_n \wedge u = \sum_{i=1}^n (x_i \wedge u)$. The sequence $(s_n \wedge u)$ is increasing and order bounded, hence is norm Cauchy by Nakano's Theorem; see [AB06, Theorem 4.9]. We can find n_0 such that $\|s_m \wedge u - s_n \wedge u\| < \varepsilon$ whenever $m \geqslant n \geqslant n_0$. Observe that

$$s_m \wedge u - s_n \wedge u = \sum_{i=n+1}^{m} (x_i \wedge u) = (s_m - s_n) \wedge u = |s_m - s_n| \wedge u.$$

It follows that $\big\||s_m - s_n| \wedge u\big\| < \varepsilon$, so that $s_m - s_n \in V_{\varepsilon,u}$. □

Proposition 6.2. *Let X be an order continuous Banach lattice. X is un-complete iff X is finite-dimensional.*

Proof. If X is finite-dimensional then it has a strong unit, so that un-topology agrees with norm topology and is, therefore, un-complete. Suppose now that $\dim X = \infty$. Then X contains a disjoint normalized positive sequence. By Lemma 6.1, X is not un-complete. □

Example 6.3. Let $X = L_p$ with $1 < p < \infty$. Pick $0 \leqslant x \in L_1 \setminus L_p$ and put $x_n = x \wedge (n\mathbb{1})$. It is easy to see that (x_n) is un-Cauchy in L_p, yet it does not un-converge in L_p.

Even when the entire space is not un-complete, the closed unit ball B_X may still be un-complete; that is, complete in the topology induced by un-topology on X. Since B_X is un-closed, it is un-complete iff every norm bounded un-Cauchy net in X is un-convergent. The following theorem should be compared with [GX14, Theorem 4.7], where a similar statement was proved for uo-convergence.

Theorem 6.4. *Let X be an order continuous Banach lattice. Then B_X is un-complete iff X is a KB-space.*

Proof. Suppose X is not KB. Then X contains a lattice copy of c_0. Let (x_n) be the sequence in X corresponding to the unit basis of c_0. Let $s_n = \sum_{i=1}^n x_i$. Clearly, (s_n) is norm bounded. However, by Lemma 6.1, (s_n) is un-Cauchy but not un-convergent.

Suppose now that X is a KB-space. First, we consider the case when X has a weak unit. In this case, un-topology on X and, therefore, on B_X, is metrizable by Theorem 3.2. Hence, it suffices to prove that B_X is sequentially un-complete. Let (x_n) be a sequence in B_X which is un-Cauchy in X. Let $L_1(\mu)$ be an AL-representation for X. It follows that (x_n) is Cauchy with respect to convergence in measure in $L_1(\mu)$. By [Fol99, Theorem 2.30], there is a subsequence (x_{n_k}) which converges a.e. It follows that (x_{n_k}) is uo-Cauchy in X by [GTX, Remark 4.6]. Then [GX14, Theorem 4.7] yields that $x_{n_k} \xrightarrow{\text{uo}} x$ for some $x \in X$. It follows that $x_{n_k} \xrightarrow{\text{un}} x$. Since (x_n) is un-Cauchy, this yields that $x_n \xrightarrow{\text{un}} x$.

Now consider the general case. Let X be a KB-space and (x_α) a net in B_X such that (x_α) is un-Cauchy in X; we need to prove that the net is un-convergent. We may assume without loss of generality that $x_\alpha \geqslant 0$ for every α; otherwise, consider (x_α^+) and (x_α^-), which are also un-Cauchy because $|x_\alpha^+ - x_\beta^+| \leqslant |x_\alpha - x_\beta|$ and $|x_\alpha^- - x_\beta^-| \leqslant |x_\alpha - x_\beta|$. By Theorem 4.11, there exists a dense band decomposition $\mathcal{B}$ of X such that each B in $\mathcal{B}$ has a weak unit. Put

$$\mathcal{C} = \bigl\{B_1 \oplus \cdots \oplus B_n \ :\ B_1, \dots, B_n \in \mathcal{B}\bigr\}.$$

Note that $\mathcal{C}$ is a family of bands with weak units. Furthermore, $\mathcal{C}$ is a directed set when ordered by inclusion, so the family of band projections $(P_C)_{C\in\mathcal{C}}$ may be viewed as a net.

For every $C \in \mathcal{C}$, the net $(P_C x_\alpha)$ is un-Cauchy by Remark 4.9. Since C has a weak unit, the first part of the proof yields that $(P_C x_\alpha)$ un-converges to some positive vector x_C in C. This produces a net $(x_C)_{C\in\mathcal{C}}$. It is easy to verify that $x_C = x_{B_1} + \cdots + x_{B_n}$ whenever $C = B_1 \oplus \cdots \oplus B_n$ for some $B_1, \dots, B_n \in \mathcal{B}$. It follows that the net $(x_C)_{C\in\mathcal{C}}$ is increasing. On the other hand, $\|x_C\| \leqslant \liminf_\alpha \|P_C x_\alpha\| \leqslant 1$, so that this net is norm bounded. Since X is a KB-space, the net $(x_C)_{C\in\mathcal{C}}$ converges in norm to some $x\in X$.

Fix $B\in\mathcal{B}$. On one hand, norm continuity of P_B yields $\lim_{C\in\mathcal{C}} P_Bx_C = P_Bx$. On the other hand, for every $C\in\mathcal{C}$ with $B\subseteq C$ we have $P_Bx_C = x_B$, so that $\lim_{C\in\mathcal{C}} P_Bx_C = x_B$. It follows that $P_Bx = x_B$, so that $P_Bx_\alpha \xrightarrow{\text{un}} P_Bx$ for every $B\in\mathcal{B}$. Now Theorem 4.12 yields $x_\alpha \xrightarrow{\text{un}} x$. □

The assumption that X is order continuous cannot be removed: for example, ℓ_∞ is not a KB-space, yet its closed unit ball is un-complete (because the un and the norm topologies on ℓ_∞ agree).

Example 6.5. The following examples show that in general B_X in Theorem 6.4 cannot be replaced with an arbitrary convex closed bounded set. Let $X=\ell_1$; let C be the set of all vectors in B_X whose coordinates sum up to zero. Clearly, C is convex, closed, and bounded. Let $x_n=\frac{1}{2}(e_1-e_n)$. Then (x_n) is a sequence in C which un-converges to $\frac{1}{2}e_1$ which is not in C. Thus, C is not un-closed in X; in particular, C is not un-complete.

It is easy to construct a similar example in $X=L_1$; take $C=\bigl\{x\in B_X : \int x=0\bigr\}$ and put $x_n=\chi_{[0,\frac{1}{2}]}-\frac{n}{2}\chi_{[\frac{1}{2},\frac{1}{2}+\frac{1}{n}]}$, $n\geqslant 2$.

Proposition 6.6. *Suppose that X^* is order continuous and C is a norm closed convex norm bounded subset of X. Then C is un-closed.*

Proof. Suppose that $x_\alpha \xrightarrow{\text{un}} x$ for a net (x_α) in C and a vector x in X. Since (x_α) is norm bounded and X^* is order continuous, [DOT, Theorem 6.4] guarantees that (x_α) converges to x weakly. Since C is convex and closed, it is weakly closed, hence $x\in C$. □

Corollary 6.7. *Let X be a reflexive Banach lattice and C a closed convex norm bounded subset of X. Then C is un-complete.*

Proof. Since X is reflexive, X is a KB-space and X^* is order continuous. Let (x_α) be a un-Cauchy net in C. Theorem 6.4 yields that $x_\alpha \xrightarrow{\text{un}} x$ for some $x\in X$, while Proposition 6.6 implies that $x\in C$. □

7. Un-compact sets

The main result of this section is Theorem 7.5, which asserts that B_X is (sequentially) un-compact iff X is an atomic KB-space. We start with some auxiliary results. The following theorem shows that, under certain assumptions, un-compactness is a "local" property.

Theorem 7.1. *Let X be a KB-space, $\mathcal{B}$ a dense band decomposition of X, and A a un-closed norm bounded subset of X. Then A is un-compact iff $P_B(A)$ is un-compact in B for every $B \in \mathcal{B}$.*

Proof. If A is un-compact then $P_B(A)$ is un-compact in B for every $B \in \mathcal{B}$ because P_B is un-continuous by Remark 4.9. To prove the converse, suppose that $P_B(A)$ is un-compact in B for every $B \in \mathcal{B}$. Let $H = \prod_{B\in\mathcal{B}} B$, the formal product of all the bands in $\mathcal{B}$. That is, H consists of families $(x_B)_{B\in\mathcal{B}}$ indexed by $\mathcal{B}$, where $x_B \in B$. We equip H with the topology of coordinate-wise un-convergence; this is the product of un-topologies on the bands that make up H. This makes H a topological vector space. Define $\Phi\colon X \to H$ via $\Phi(x) = (P_Bx)_{B\in\mathcal{B}}$. Clearly, Φ is linear. Since $\mathcal{B}$ is a dense band decomposition, Φ is one-to-one. By Theorem 4.12, Φ is a homeomorphism from X equipped with un-topology onto its range in H.

Let K be the subset of H defined by $K = \prod_{B\in\mathcal{B}} P_B(A)$. By Tikhonov's Theorem, K is compact in H. It is easy to see that $\Phi(A) \subseteq K$.

We claim that $\Phi(A)$ is closed in H. Indeed, suppose that $\Phi(x_\alpha) \to h$ in H for some net (x_α) in A. In particular, the net $\big(\Phi(x_\alpha)\big)$ is Cauchy in H. Since Φ is a homeomorphism, the net (x_α) is un-Cauchy in A. Since (x_α) is bounded and X is a KB-space, (x_α) un-converges to some $x \in X$ by Theorem 6.4. Since A is un-closed, we have $x \in A$. It follows that $h = \Phi(x)$, so that $h \in \Phi(A)$.

Being a closed subset of a compact set, $\Phi(A)$ is itself compact. Since Φ is a homeomorphism, we conclude that A is un-compact. □

Next, we discuss relationships between the sequential and the general variants of un-closedness and un-compactness. Recall that for a set A in a topological space, we write $\overline{A}$ for the closure of A; we write $\overline{A}^\sigma$ for the ***sequential closure*** of A, i.e., $a \in \overline{A}^\sigma$ iff a is the limit of a sequence in A. We say that A is ***sequentially closed*** if $\overline{A}^{\sigma} = A$. It is well known that for a metrizable topology, we always have $\overline{A}^{\sigma} = \overline{A}$.

For a set A in a Banach lattice, we write $\overline{A}^{\mathrm{un}}$ and $\overline{A}^{\sigma\text{-un}}$ for the un-closure and the sequential un-closure of A, respectively. Obviously, $\overline{A}^{\sigma\text{-un}} \subseteq \overline{A}^{\mathrm{un}}$.

Example 7.2. In general, $\overline{A}^{\mathrm{un}} \neq \overline{A}^{\sigma\text{-un}}$. Indeed, in the notation of Example 1.3, let $A = \{e_\omega : \omega \in \Omega\}$. It follows from Example 1.3 that zero is in $\overline{A}^{\mathrm{un}}$ but not in $\overline{A}^{\sigma\text{-un}}$.

Proposition 7.3. *Let A be a subset of a Banach lattice X. If X has a quasi-interior point* or *X is order continuous then $\overline{A}^{\mathrm{un}} = \overline{A}^{\sigma\text{-un}}$.*

Proof. If X has a quasi-interior point then its un-topology is metrizable by Theorem 3.2, hence $\overline{A}^{\mathrm{un}} = \overline{A}^{\sigma\text{-un}}$.

Suppose that X is order continuous. Suppose that $x \in \overline{A}^{\mathrm{un}}$; we need to show that $x \in \overline{A}^{\sigma\text{-un}}$. Without loss of generality, $x = 0$. This means that A contains a un-null net (x_α). By Theorem 1.1, there exists an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \xrightarrow{\|\cdot\|} 0$. It follows that $x_{\alpha_k} - d_k \xrightarrow{\mathrm{un}} 0$. Since (d_k) is disjoint, it is uo-null and, since X is order continuous, un-null. It follows that $x_{\alpha_k} \xrightarrow{\mathrm{un}} 0$ and, therefore, $0 \in \overline{A}^{\sigma\text{-un}}$. □

Recall that a topological space is said to be ***sequentially compact*** if every sequence has a convergent subsequence. In a Hausdorff topological vector space which is metrizable (or, equivalently, first countable), sequential compactness is equivalent to compactness, see, e.g., [Roy88, Theorem 7.21]. We do not know whether un-compactness and sequential un-compactness are equivalent in general, yet we have the following partial result.

Proposition 7.4. *Let A be a subset of a Banach lattice X.*

(i) *If X has a quasi-interior point, then A is sequentially un-compact iff A is un-compact.*
(ii) *Suppose that X is order continuous. If A is un-compact then A is sequentially un-compact.*
(iii) *Suppose that X is a KB-space. If A is norm bounded and sequentially un-compact then A is un-compact.*

Proof. (i) follows immediately from Theorem 3.2.

(ii) Let (x_n) be a sequence in A. Find $e \in X_+$ such that (x_n) is contained in B_e (e.g., take $e = \sum_{n=1}^{\infty} \frac{x_n}{2^n\|x_n\|+1}$). Since B_e is un-closed, the set $A \cap B_e$ is un-compact in B_e. Since e is a quasi-interior point for B_e, the un-topology on B_e is metrizable, hence $A \cap B_e$ is sequentially un-compact. It follows that there is a subsequence (x_{n_k}) which un-converges in B_e to some $x \in A \cap B_e$. By Theorem 4.3(iii), $x_{n_k} \xrightarrow{\text{un}} x$ in X.

(iii) Clearly, A is sequentially un-closed and, therefore, un-closed by Proposition 7.3. Let $\mathcal{B}$ be as in Theorem 4.11. For each $B \in \mathcal{B}$, the band projection P_B is un-continuous by Remark 4.9, so that $P_B(A)$ is sequentially un-compact in B. Since B has a weak unit, the un-topology on B is metrizable, so that $P_B(A)$ is un-compact in B. The conclusion now follows from Theorem 7.1. □

Theorem 7.5. *For a Banach lattice X, TFAE:*

(i) *B_X is un-compact;*
(ii) *B_X is sequentially un-compact;*
(iii) *X is an atomic KB-space.*

Proof. First, observe that both (i) and (ii) imply that X is order continuous and atomic. Indeed, since order intervals are bounded and un-closed, they are (sequentially) un-compact. But on order intervals, the un-topology agrees with the norm topology, hence order intervals are norm compact. This implies that X is atomic and order continuous; see, e.g., [Wnuk99, Theorem 6.1].

Suppose (i). Since X is order continuous, Proposition 7.4(ii) yields (ii).

Suppose (ii). We already know that X is atomic. To show that X is a KB-space, let (x_n) be an increasing norm bounded sequence in X_+. By assumption, it has a un-convergent subsequence (x_{n_k}). By Lemma 1.2(ii), (x_{n_k}) converges in norm, hence (x_n) converges in norm. This yields (iii).

Suppose (iii). Let A be a maximal disjoint family of atoms in X. Then $\{B_a : a \in A\}$ is a dense band decomposition of X. For every $a \in A$, $P_a(B_X)$ is a closed bounded subset of the one-dimensional band

B_a, hence $P_a(B_X)$ is norm and un-compact in B_a. Theorem 7.1 now implies that B_X is un-compact, which yields (i). □

Example 7.6. Let $X = c_0$ and $x_n = e_1 + \dots + e_n$. Then (x_n) is a sequence in B_X with no un-convergent subsequences.

Proposition 7.7. *Let A be a subset of an order continuous Banach lattice X. If A is relatively un-compact then A is relatively sequentially un-compact.*

Proof. Let (x_n) be a sequence in A. Find $e \in X_+$ such that (x_n) is contained in B_e. Since $\overline{A}^{\mathrm{un}}$ is un-compact, the set $\overline{A}^{\mathrm{un}} \cap B_e$ is un-compact in B_e and, therefore, sequentially un-compact in B_e because the un-topology on B_e is metrizable. Hence, there is a subsequence (x_{n_k}) which un-converges in B_e and, therefore, in X. □

8. Un-convergence and weak*-convergence

When does un-convergence imply weak*-convergence? It is easy to see that, in general, un-convergence does not imply weak*-convergence. Indeed, let X be an infinite-dimensional Banach lattice with order continuous dual. Pick any unbounded disjoint sequence (f_n) in X^*. Being unbounded, (f_n) cannot be weak*-null. Yet it is un-null by Proposition 3.5. However, if we restrict ourselves to norm bounded nets, the situation is more interesting. The following result is analogous to [Gao14, Theorem 2.1]. Recall that for a net (f_α) in X^*, we write $f_\alpha \xrightarrow{|\sigma|(X^*,X)} 0$ if $|f_\alpha|(x) \to 0$ for every $x \in X_+$.

Theorem 8.1. *Let X be a Banach lattice such that X^* is order continuous. The following are equivalent:*

(i) *X is order continuous;*
(ii) *for any norm bounded net (f_α) in X^*, if $f_\alpha \xrightarrow{\mathrm{un}} 0$, then $f_\alpha \xrightarrow{\mathrm{w}^*} 0$;*
(iii) *for any norm bounded net (f_α) in X^*, if $f_\alpha \xrightarrow{\mathrm{un}} 0$, then $f_\alpha \xrightarrow{|\sigma|(X^*,X)} 0$;*
(iv) *for any norm bounded sequence (f_n) in X^*, if $f_n \xrightarrow{\mathrm{un}} 0$, then $f_n \xrightarrow{\mathrm{w}^*} 0$;*

(v) *for any norm bounded sequence* (f_n) *in* X^**, if* $f_n \xrightarrow{\mathrm{un}} 0$*, then* $f_n \xrightarrow{|\sigma|(X^*,X)} 0$.

The proof is similar to that of [Gao14, Theorem 2.1] except that in the proof of (iv)⇒(i) we use Proposition 3.5. Note that without the assumption that X^* is order continuous, we still get the following implications:

$$\text{(i)} \Rightarrow \big[\text{(ii)} \Leftrightarrow \text{(iii)}\big] \Rightarrow \big[\text{(iv)} \Leftrightarrow \text{(v)}\big].$$

When does weak*-convergence imply un-convergence? Recall that for norm bounded nets, weak*-convergence implies uo-convergence in X^* iff X is atomic and order continuous by [Gao14, Theorem 3.4]. Furthermore, Proposition 4.16 immediately yields the following.

Corollary 8.2. *If* $f_n \xrightarrow{\mathrm{w}^*} 0$ *implies* $f_n \xrightarrow{\mathrm{un}} 0$ *for every sequence in* X^* *then* X^* *is atomic and order continuous.*

The following example shows that the converse is false in general.

Example 8.3. Let $X = c$, the space of all convergent sequences. By [AB06a, Theorem 16.14], X^* may be identified with $\ell_1 \oplus \mathbb{R}$ with the duality given by

$$\big\langle (f,r), x \big\rangle = r \cdot \lim_n x_n + \sum_{n=1}^{\infty} f_n x_n,$$

where $x \in c$, $f \in \ell_1$, and $r \in \mathbb{R}$. It is easy to see that X^* is atomic and order continuous. Consider the sequence $\big((e_n, 0)\big)$ in X^*, where e_n is the n-th standard unit vector in ℓ_1. It is easy to see that $(e_n, 0) \xrightarrow{\mathrm{w}^*} (0, 1)$ in X^*. On the other hand, this sequence is disjoint and, therefore, un-null. Take $f_n = (e_n, -1)$; it follows that (f_n) is weak*-null but not un-null. Note that in this example, X^* is order continuous while X is not.

Nevertheless, we will show that the converse implication is true under the additional assumption that X is order continuous.

Theorem 8.4. *The following are equivalent:*

(i) *For every net* (f_α) *in* X^**, if* $f_\alpha \xrightarrow{\mathrm{w}^*} 0$ *then* $f_\alpha \xrightarrow{\mathrm{un}} 0$*;*

(ii) X^* *is atomic and both* X *and* X^* *are order continuous.*

Proof. (i)⇒(ii) By Corollary 8.2, X^* is atomic and order continuous. Suppose X is not order continuous. By [MN91, Corollary 2.4.3] there exists a disjoint norm-bounded sequence (f_n) in X^* which is not weak*-null. One can then find a subsequence (f_{n_k}), a vector $x_0 \in X$ and a positive real ε so that $\lvert f_{n_k}(x_0)\rvert > \varepsilon$ for every k. By the Alaoglu-Bourbaki Theorem, there is a subnet (g_α) of (f_{n_k}) such that $g_\alpha \xrightarrow{\mathrm{w}^*} g$ for some $g \in X^*$. Since (f_{n_k}) is disjoint and X^* is order continuous, we have $f_{n_k} \xrightarrow{\mathrm{un}} 0$ and, therefore, $g_\alpha \xrightarrow{\mathrm{un}} 0$. By assumption, this yields $g = 0$, so that $g_\alpha \xrightarrow{\mathrm{w}^*} 0$. This contradicts $\lvert g_\alpha(x_0)\rvert > \varepsilon$ for every α.

(ii)⇒(i) Let $f_\alpha \xrightarrow{\mathrm{w}^*} 0$ in X. Let A be a maximal disjoint collection of atoms in X^*; for each atom $a \in A$ let P_a and φ_a be the corresponding band projection and the coordinate functional, respectively; P_a and φ_a are defined on X^*. By [MN91, Corollary 2.4.7], P_a and, therefore, φ_a, is weak*-continuous. It follows that $\varphi_a(f_\alpha) \to 0$ in α. Corollary 4.14 yields that $f_\alpha \xrightarrow{\mathrm{un}} 0$. □

Proposition 8.5. *Suppose that X^* is atomic. The following are equivalent.*

(i) *For every net (f_α) in X^*, if $f_\alpha \xrightarrow{\lvert\sigma\rvert(X^*,X)} 0$ then $f_\alpha \xrightarrow{\mathrm{un}} 0$;*

(ii) *For every sequence (f_n) in X^*, if $f_n \xrightarrow{\lvert\sigma\rvert(X^*,X)} 0$ then $f_n \xrightarrow{\mathrm{un}} 0$;*

(iii) *X^* is order continuous.*

Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii) The proof is similar to that of Proposition 4.16. To show that X^* is order continuous, suppose that (f_n) is an order bounded positive disjoint sequence in X^*_+. It follows that $f_n \xrightarrow{\lvert\sigma\rvert(X^*,X)} 0$ and, by assumption, $f_n \xrightarrow{\mathrm{un}} 0$. Since the sequence is order bounded, this yields $f_n \xrightarrow{\lVert\cdot\rVert} 0$. Therefore, X^* is order continuous.

(iii)⇒(i) By [MN91, Proposition 2.4.5], band projections on X^* are $\lvert\sigma\rvert(X^*,X)$-continuous. The proof is now analogous to the implication (ii)⇒(i) in Theorem 8.4. □

Simultaneous weak* and un-convergence. Section 4 of [Gao14] contains several results that assert that if a sequence or a net in X^* converges in both weak* and uo-topology then it also converges in some other topology. Several of these results remain valid if uo-convergence is replaced with un-convergence. In particular, this works for Proposition 4.1 in [Gao14]. Propositions 4.3, 4.4, and 4.6 in [Gao14] remain valid under the additional assumption that X^* is order continuous (note that the dual positive Schur property already implies that X^* is order continuous by [Wnuk13, Proposition 2.1]). The proofs are analogous to the corresponding proofs in [Gao14]. Alternatively, the un-versions of these may be deduced from the uo-versions using the following two facts: first, every un-convergent sequence has a uo-convergent subsequence and, second, a sequence (x_n) converges to x in a topology τ iff every subsequence (x_{n_k}) has a further subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \xrightarrow{\tau} x$.

9. Un-compact operators

Throughout this section, let E be a Banach space, X a Banach lattice, and $T \in L(E, X)$. We say that T is ***(sequentially) un-compact*** if TB_E is relatively (sequentially) un-compact in E. Equivalently, for every bounded net (x_α) (respectively, every bounded sequence (x_n)) its image has a subnet (respectively, subsequence), which is un-convergent.

Clearly, if T is compact then it is un-compact and sequentially un-compact. Theorems 3.2 and 7.5 and Proposition 7.7 yield the following.

Proposition 9.1. *Let $T \in L(E, X)$.*

- (i) *If X has a quasi-interior point then T is un-compact iff it is sequentially un-compact;*
- (ii) *If X is order continuous and T is un-compact then T is sequentially un-compact;*
- (iii) *If X is an atomic KB-space then T is un-compact and sequentially un-compact.*

Proposition 9.2. *The set of all un-compact operators is a linear subspace of $L(E, X)$. The set of all sequentially un-compact operators in $L(E, X)$ is a closed subspace.*

Proof. Linearity is straightforward. To prove closedness, suppose that (T_m) is a sequence of sequentially un-compact operators in $L(E, X)$ and $T_m \xrightarrow{\|\cdot\|} T$. We will show that T is sequentially un-compact.

Let (x_n) be a sequence in B_E. For every m, the sequence $(T_m x_n)_n$ has a un-convergent subsequence. By a standard diagonal argument, we can find a common subsequence for all these sequences. Passing to this subsequence, we may assume without loss of generality that for every m we have $T_m x_n \xrightarrow{\mathrm{un}} y_m$ for some y_m. Note that

$$\|y_m - y_k\| \leqslant \liminf_n \|T_m x_n - T_k x_n\| \leqslant \|T_m - T_k\| \to 0,$$

so that the sequence (y_m) is Cauchy and, therefore, $y_m \xrightarrow{\|\cdot\|} y$ for some $y \in X$.

Fix $u \in X_+$ and $\varepsilon > 0$. Find m_0 such that $\|T_{m_0} - T\| < \varepsilon$ and $\|y_{m_0} - y\| < \varepsilon$. Find n_0 such that $\big\| |T_{m_0} x_n - y_{m_0}| \wedge u \big\| < \varepsilon$ whenever $n \geqslant n_0$. It follows from

$$|T x_n - y| \wedge u \leqslant |T x_n - T_{m_0} x_n| + |T_{m_0} x_n - y_{m_0}| \wedge u + |y_{m_0} - y|$$

that $\big\| |T x_n - y| \wedge u \big\| < 3\varepsilon$, so that $T x_n \xrightarrow{\mathrm{un}} y$. □

We do not know whether the set of all un-compact operators is closed.

It is easy to see that if we multiply a (sequentially) un-compact operator by another bounded operator on the right, the product is again (sequentially) un-compact. The following example shows that this fails when we multiply on the left.

Example 9.3. *The class of all (sequentially) un-compact operators is not a left ideal.* Let $T\colon \ell_1 \to L_1$ be defined via $T e_n = r_n^+$, where (e_n) is the standard unit basis of ℓ_1 and (r_n) is the Rademacher sequence in L_1. Note that T is neither un-compact nor sequentially un-compact because the sequence $(T e_n)$ has no un-convergent subsequences. On the other hand, $T = T I_{\ell_1}$, where I_{ℓ_1} is the identity operator on ℓ_1. Observe that I_{ℓ_1} is un-compact by Proposition 9.1(iii).

Proposition 9.4. *In the diagram $E \xrightarrow{T} X \xrightarrow{S} Y$, suppose that T is (sequentially) un-compact and S is a lattice homomorphism. If the ideal generated by* Range S *is dense in Y then ST is (sequentially) un-compact.*

Proof. We will prove the statement for the sequential case; the other case is analogous. Let (h_n) be a norm bounded sequence in E. By assumption, there is a subsequence (h_{n_k}) such that $Th_{n_k} \xrightarrow{\text{un}} x$ for some $x \in X$. Let $Z = \operatorname{Range} S$; it is a sublattice of Y. Fix $u \in Z_+$. Then $u = Sv$ for some $v \in X_+$, and $|Th_{n_k} - x| \wedge v \xrightarrow{\|\cdot\|} 0$. Applying S, we get $\big|STh_{n_k} - Sy\big| \wedge u \xrightarrow{\|\cdot\|} 0$. Therefore, $STh_{n_k} \xrightarrow{\text{un}} Sx$ in Z. It follows from Theorem 4.3(i) and (ii) that $STh_{n_k} \xrightarrow{\text{un}} Sx$ in Y. □

Example 9.5. *The set of all sequentially un-compact operators is not order closed.* Let T be as in Example 9.3. Let $T_n = TP_n$, where P_n is the n-th basis projection on ℓ_1, i.e., $T_nh = \sum_{i=1}^n h_i r_i^+$ for $h \in \ell_1$. It is easy to see that each T_n is finite rank and, therefore, sequentially un-compact. Note that $T_n \uparrow T$, yet T is not sequentially un-compact.

Proposition 9.6. *Suppose that for every sequence (T_n) of sequentially un-compact operators in $L(c_0, X)$, $T_n \uparrow T$ implies that T is sequentially un-compact. Then X is a KB-space.*

Proof. Suppose not. Then there is a lattice isomorphism $T\colon c_0 \to X$. Put $x_n = Te_n$, where (e_n) is the standard unit basis of c_0. Put $T_n = TP_n$, where P_n is the n-th basis projection on c_0, i.e., $T_nh = \sum_{i=1}^n h_i x_i$ for $h \in c_0$. It follows that $T_nh \xrightarrow{\|\cdot\|} Th$, so that $T_nh \uparrow Th$ for every $h \geqslant 0$ and, therefore, $T_n \uparrow T$. For each n, T_n has finite rank and, therefore, is sequentially un-compact.

We claim that, nevertheless, T is not sequentially un-compact. Put $w_n = e_1 + \dots + e_n$ in c_0. Note that (w_n) is norm bounded and $Tw_n = x_1 + \dots + x_n$. Since T is an isomorphism, (Tw_n) is not norm-convergent. Since (Tw_n) is increasing, it is not un-convergent by Lemma 1.2(ii). Similarly, no subsequence of (Tw_n) is un-convergent. □

We do not know whether the converse is true.

Next, we study whether un-compactness is inherited under domination. The following example shows that, in general, the answer is negative.

Example 9.7. Let T be as in Example 9.3. Let $S\colon \ell_1 \to L_1$ be defined via $Se_n = \mathbb{1}$. Then S is a rank-one operator; hence it is compact and un-compact. Clearly, $0 \leqslant T \leqslant S$. Yet T is not un-compact.

Proposition 9.8. *Suppose that $S, T\colon E \to X$, $0 \leqslant S \leqslant T$, X is a KB-space and T is a lattice homomorphism. If T is (sequentially) un-compact then so is S.*

Proof. We will prove the sequential case; the other case is similar. Let (h_n) be a bounded sequence in E. Passing to a subsequence, we may assume that (Th_n) is un-convergent. In particular, it is un-Cauchy. Fix $u \in X_+$. Note that

$$|Sh_n - Sh_m| \wedge u \leqslant \big(S|h_n - h_m|\big) \wedge u \leqslant \big(T|h_n - h_m|\big) \wedge u = |Th_n - Th_m| \wedge u \xrightarrow{\|\cdot\|} 0$$

as $n, m \to \infty$. It follows that (Sh_n) is un-Cauchy and, therefore, un-converges by Theorem 6.4. □

We would like to mention that the class of un-compact operators is different from several other known classes of operators. We already mentioned that every compact operator is un-compact. The converse is false as the identity operator on any infinite-dimensional atomic KB-space is un-compact but not compact.

Recall that an operator between Banach lattices is AM-compact if it maps order intervals to relatively compact sets.

Proposition 9.9. *Every order bounded un-compact operator is AM-compact.*

Proof. Let $T\colon X \to Y$ be an order bounded un-compact operator between Banach lattices. Fix an order interval $[a, b]$ in X. Since T is un-compact, $T[a, b] \subseteq C$ for some un-compact set C. Since T is order bounded, $T[a, b] \subseteq [c, d]$ for some $c, d \in Y$. Note that $[c, d]$ is un-closed, hence $C \cap [c, d]$ is un-compact and, being order bounded, is compact. It follows that $T[a, b]$ is relatively compact. □

Note that the converse is false: the identity operator on c_0 is AM-compact but not un-compact.

The identity operator on ℓ_1 is un-compact, yet it is neither L-weakly compact nor M-weakly compact.

Finally, we note that if T is sequentially un-compact and semi-compact then T is compact. Indeed, let (h_n) be a bounded sequence in E. There is a subsequence (h_{n_k}) such that $Th_{n_k} \xrightarrow{\text{un}} x$ for some $x \in X$.

Since T is semi-compact, the sequence (Th_{n_k}) is almost order bounded and, therefore, $Th_{n_k} \xrightarrow{\|\cdot\|} x$ by [DOT, Lemma 2.9].

Finally, we discuss when weakly compact operators are un-compact.

Lemma 9.10. *If $x_n \xrightarrow{\mathrm{w}} x$ and $x_n \xrightarrow{\mathrm{un}} y$ then $x = y$.*

Proof. Without loss of generality, $y = 0$. By Theorem 1.1, there exist a subsequence (x_{n_k}) and a disjoint sequence (d_k) such that $x_{n_k} - d_k \xrightarrow{\|\cdot\|} 0$. It follows that $x_{n_k} - d_k \xrightarrow{\mathrm{w}} 0$, so that $d_k \xrightarrow{\mathrm{w}} x$. Now [AB06, Theorem 4.34] yields $x = 0$. □

Theorem 9.11. *A Banach lattice X is atomic and order continuous iff T is sequentially un-compact for every Banach space E and every weakly compact operator $T\colon E \to X$.*

Proof. The forward implication follows immediately from Proposition 4.16. To prove the converse, let (x_n) be a weakly null sequence in X. By Proposition 4.16, it suffices to show that $x_n \xrightarrow{\mathrm{un}} 0$. Define $T\colon \ell_1 \to X$ via $Te_n = x_n$. By [AB06, Theorem 5.26], T is weakly compact. By assumption, T is sequentially un-compact. It follows that (Te_n) has a un-convergent subsequence, i.e., $x_{n_k} \xrightarrow{\mathrm{un}} x$ for some $x \in X$ and a subsequence (x_{n_k}). Lemma 9.10 yields $x = 0$. By the same argument, every subsequence of (x_n) has a further subsequence which is un-null; since un-convergence is topological, it follows that $x_n \xrightarrow{\mathrm{un}} 0$. □

Corollary 9.12. *Every operator from a reflexive Banach space to an atomic order continuous Banach lattice is sequentially un-compact.*

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