UNBOUNDED NORM CONVERGENCE
IN BANACH LATTICES

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ABSTRACT. A net (x_α) in a vector lattice X is unbounded order convergent to $x \in X$ if $|x_\alpha - x| \wedge u$ converges to 0 in order for all $u \in X_+$. This convergence has been investigated and applied in several recent papers by Gao et al. It may be viewed as a generalization of almost everywhere convergence to general vector lattices. In this paper, we study a variation of this convergence for Banach lattices. A net (x_α) in a Banach lattice X is unbounded norm convergent to x if $\||x_\alpha - x| \wedge u\| \to 0$ for all $u \in X_+$. We show that this convergence may be viewed as a generalization of convergence in measure. We also investigate its relationship with other convergences.

1. INTRODUCTION

We begin by recalling a few definitions. A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is said to be ***order convergent*** to $x \in X$ if there is a net $(z_\beta)_{\beta \in B}$ in X such that $z_\beta \downarrow 0$ and for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq z_\beta$ whenever $\alpha \geq \alpha_0$. For short, we will denote this convergence by $x_\alpha \xrightarrow{o} x$ and write that x_α is o-convergent to x. A net $(x_\alpha)_{\alpha \in A}$ in a vector lattice X is ***unbounded order convergent*** to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in X_+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that x_α uo-converges to x. We refer the reader to [GTX] for a detailed exposition on uo-convergence and further references. In particular, for sequences in $L_p(\mu)$ spaces, uo-convergence agrees with almost everywhere (a.e.) convergence. Furthermore, if X can be represented as an ideal (or, more generally, a regular sublattice) in $L_1(\mu)$, then the uo-convergence of sequences in X agrees with the a.e. convergence in $L_1(\mu)$. Thus, uo-convergence may be viewed as a generalization of a.e. convergence to general vector lattices.

Throughout this paper, X will stand for a Banach lattice. For a net (x_α) in X we write $x_\alpha \to x$ if (x_α) converges to x in norm, i.e., $\|x_\alpha - x\| \to 0$. A net (x_α) in X is **unbounded norm convergent** or **un-convergent** to x if $|x_\alpha - x| \wedge u \to 0$ for all $u \in X_+$; we then write $x_\alpha \xrightarrow{\text{un}} x$. Un-convergence was introduced in [Tro04] as a tool to study measures of non-compactness. In this paper, we study properties of un-convergence and its relationship to other convergences. In particular, we show that un-convergence may be viewed as a generalization of convergence in measure to general Banach lattices.

2. BASIC PROPERTIES OF UN-CONVERGENCE

Unless stated otherwise, we will assume that X is a Banach lattice and all nets and vectors lie in X. We routinely use the following inequality: $(x + y) \wedge u \le x \wedge u + y \wedge u$ for all $x, y, u \in X_+$; see, e.g., [AB06, Lemma 1.4].

Lemma 2.1.   (i) $x_\alpha \xrightarrow{\text{un}} x$ *iff* $(x_\alpha - x) \xrightarrow{\text{un}} 0$;
   (ii) *If* $x_\alpha \xrightarrow{\text{un}} x$, *then* $y_\beta \xrightarrow{\text{un}} x$ *for any subnet* (y_β) *of* (x_α).
  (iii) *Suppose* $x_\alpha \xrightarrow{\text{un}} x$ *and* $y_\alpha \xrightarrow{\text{un}} y$. *Then* $ax_\alpha + by_\alpha \xrightarrow{\text{un}} ax + by$ *for any* $a, b \in \mathbb{R}$.
  (iv) *If* $x_\alpha \xrightarrow{\text{un}} x$ *and* $x_\alpha \xrightarrow{\text{un}} y$, *then* $x = y$.
  (v) *If* $x_\alpha \xrightarrow{\text{un}} x$, *then* $|x_\alpha| \xrightarrow{\text{un}} |x|$.

Proof. (i), (ii), and (iii) are straightforward. To prove (iv), observe that $|x - y| \le |x - x_\alpha| + |y - x_\alpha|$ for every α. Put $u = |x - y|$; it follows that

$$|x - y| = |x - y| \wedge u \le |x - x_\alpha| \wedge u + |y - x_\alpha| \wedge u \to 0.$$

Finally, (v) follows from $\big||x_\alpha| - |x|\big| \le |x_\alpha - x|$.   □

Remark 2.2. In particular, $x_\alpha \xrightarrow{\text{un}} x$ iff $|x_\alpha - x| \xrightarrow{\text{un}} 0$. This often allows one to reduce general un-convergence to un-convergence of positive nets to zero.

Example 2.3. It was observed in Examples 21 and 22 in [Tro04] that on c_0 un-convergence agrees with coordinate-wise convergence and on $C_0(\Omega)$ un-convergence agrees with uniform convergence on compacta.

The proofs of the following two facts are straightforward.

Proposition 2.4. *If* $x_\alpha \to 0$ *then* $x_\alpha \xrightarrow{\text{un}} 0$. *For order bounded nets, un-convergence and norm convergence agree.*

This justifies the name *unbounded* norm convergence.

Proposition 2.5. *In an order continuous Banach lattice, uo-convergence implies un-convergence.*

Example 2.6. Let (e_n) be the standard unit sequence in ℓ_∞. It follows immediately from Proposition 2.4 that it is not un-null. This serves as a counterexample to several "natural" statements. First, while [GTX, Corollary 3.6] asserts that every disjoint sequence is uo-null (see also [Gao14, Lemma 1.1]), this example shows that a disjoint sequence need not be un-null. Second, since (e_n) is uo-null, this example shows that the order continuity assumption in Proposition 2.5 cannot be dropped.

Third, it was observed in [GTX, Theorem 3.2] that for a net (x_α) in a regular sublattice F of a vector lattice E, $x_\alpha \xrightarrow{\text{uo}} 0$ in F iff $x_\alpha \xrightarrow{\text{uo}} 0$ in E. This fails for un-convergence: indeed, (e_n) is un-null as a sequence in c_0, but not in ℓ_∞. However, it is easy to see that if $x_\alpha \xrightarrow{\text{un}} 0$ in X then $x_\alpha \xrightarrow{\text{un}} 0$ in every sublattice of X.

Example 2.7. The next example shows that un-convergence in a sublattice does not imply un-convergence in the entire space even when the sublattice is a lattice copy of ℓ_1. Let $X = \ell_1 \oplus_\infty \ell_\infty$; let (f_n) be the standard unit basis of ℓ_1 and (g_n) the standard unit sequence in ℓ_∞. Put $x_n = f_n \oplus g_n$. Let Y be the closed span of (x_n) in X. Since (x_n) is a disjoint sequence in X, Y is exactly the closed sublattice generated by (x_n). Observe that

$$\left\|\sum_{k=1}^n \alpha_k x_k\right\| = \left\|\sum_{k=1}^n \alpha_k f_k\right\| \vee \left\|\sum_{k=1}^n \alpha_k g_k\right\| = \left(\sum_{k=1}^n |\alpha_k|\right) \vee \left(\bigvee_{k=1}^n |\alpha_k|\right) = \sum_{k=1}^n |\alpha_k|.$$

for any n and any scalars $\alpha_1, \ldots, \alpha_n$. It follows that the basic sequence (x_n) in X is 1-equivalent to (f_n) in ℓ_1 and, therefore, Y is an isometric lattice copy of ℓ_1 in X. It is easy to see that $f_n \xrightarrow{\text{un}} 0$ in ℓ_1; hence $x_n \xrightarrow{\text{un}} 0$ in Y.

However, we claim that $x_n \xrightarrow{\text{un}} \!\!\!\!\!/\,\, 0$ in X. Indeed, let $u = 0 \oplus \mathbb{1} = \bigvee_{k=1}^\infty g_k$. Then $x_n \wedge u = g_n$ for every n, hence $(x_n \wedge u)$ does not converge to zero in X.

The following three results are similar to Lemmas 3.6 and 3.7, and Proposition 3.9 of [GX14]; we replace uo-convergence with un-convergence, and we do not require the space be order continuous in this case. The proofs are similar.

Lemma 2.8. *If $x_\alpha \xrightarrow{\text{un}} x$ then $|x_\alpha| \wedge |x| \to |x|$ and $\|x\| \le \liminf_\alpha \|x_\alpha\|$.*

Recall that a subset A of X is **_almost order bounded_** if for every $\varepsilon > 0$ there exists $u \in X_+$ such that $A \subseteq [-u, u] + \varepsilon B_X$. Equivalently, $\left\|(|x| - u)^+\right\| < \varepsilon$ for all $x \in A$.

Lemma 2.9. *If $x_\alpha \xrightarrow{\text{un}} x$ and (x_α) is almost order bounded then $x_\alpha \to x$.*

Proposition 2.10. *If (x_α) is relatively weakly compact and $x_\alpha \xrightarrow{\text{un}} x$ then (x_α) converges to x in $|\sigma|(X, X^*)$.*

Proof. Without loss of generality, $x = 0$. Let $f \in X_+^*$. Fix $\varepsilon > 0$. By [AB06, Theorem 4.37], there exists $u \in X_+$ such that $f\left(\left(|x_\alpha| - u\right)^+\right) < \varepsilon$ for every α. It follows from $|x_\alpha| \wedge u \to 0$ that $f(|x_\alpha| \wedge u) \to 0$, so that

$$f\left(|x_\alpha|\right) = f\left(|x_\alpha| \wedge u\right) + f\left(\left(|x_\alpha| - u\right)^+\right) < 2\varepsilon$$

for all sufficiently large α. It follows that $f\left(|x_\alpha|\right) \to 0$. $\square$

If (x_α) is a net in a vector lattice with a weak unit e then $x_\alpha \xrightarrow{\text{uo}} 0$ iff $|x_\alpha| \wedge e \xrightarrow{\text{o}} 0$; see, e.g., [GTX, Lemma 3.5]. Analogously, the next result limits the task of checking un-convergence to a single quasi-interior point, if one exists; cf [Tro04, Lemma 24].

Lemma 2.11. *Let X be a Banach lattice with a quasi-interior point e. Then $x_\alpha \xrightarrow{\text{un}} 0$ iff $|x_\alpha| \wedge e \to 0$.*

Proof. The forward implication is immediate. For the reverse implication, let $u \in X_+$ be arbitrary and fix $\varepsilon > 0$. Note that

$$|x_\alpha| \wedge u \le |x_\alpha| \wedge (u - u \wedge me) + |x_\alpha| \wedge (u \wedge me) \le (u - u \wedge me) + m\left(|x_\alpha| \wedge e\right)$$

and, therefore,

$$\left\| |x_\alpha| \wedge u \right\| \le \|u - u \wedge me\| + m\left\| |x_\alpha| \wedge e \right\|$$

for all α and all $m \in \mathbb{N}$. Since e is quasi-interior, we can find m such that $\|u - u \wedge me\| < \varepsilon$. Furthermore, it follows from $|x_\alpha| \wedge e \to 0$ that there exists α_0 such that $\left\| |x_\alpha| \wedge e \right\| < \frac{\varepsilon}{m}$ whenever $\alpha \ge \alpha_0$. It follows that $\left\| |x_\alpha| \wedge u \right\| < \varepsilon + m\frac{\varepsilon}{m} = 2\varepsilon$. Therefore, $|x_\alpha| \wedge u \to 0$. $\square$

Corollary 2.12. *Let X be an order continuous Banach lattice with a weak unit e. Then $x_\alpha \xrightarrow{\text{un}} 0$ iff $|x_\alpha| \wedge e \to 0$.*

Proof. If X is order continuous, then e is a weak unit iff e is a quasi-interior point. $\square$

Recall that norm convergence is sequential in nature. In particular, given a net (x_α) in a normed space, if $x_\alpha \to x$ then there exists an increasing sequence of indices (α_n) such that $x_{\alpha_n} \to x$. This often allows one to reduce nets to sequences when dealing with norm convergence. In view of Lemma 2.11, we can do the same with the un-convergence

as long as the space has a quasi-interior point (in particular, when X is separable):

Corollary 2.13. *Suppose that X has a quasi-interior point and $x_\alpha \xrightarrow{\text{un}} 0$ for some net (x_α) in X. Then there exists an increasing sequence of indices (α_n) such that $x_{\alpha_n} \xrightarrow{\text{un}} 0$.*

Question 2.14. Does Corollary 2.13 remain valid without a quasi-interior point?

We will show in Corollary 3.5 that the answer is affirmative for order continuous spaces.

3. DISJOINT SUBSEQUENCES

The following lemma is standard; we provide the proof for the convenience of the reader.

Lemma 3.1. *Let $|x| = u + v$ for some vector x and some positive vectors u and v in a vector lattice. Then there exist y and z such that $x = y + z$, $|y| = u$, and $|z| = v$.*

Proof. Applying the Riesz Decomposition Property [AB06, Theorem 1.20] to the equality $x^+ + x^- = u + v$, we find four positive vectors vectors a, b, c, and d such that $u = a + b$, $v = c + d$, $x^+ = a + c$, and $x^- = b + d$. Put $y = a - b$ and $z = c - d$. Then $y + z = x^+ - x^- = x$. It follows from $0 \le a \le x^+$ and $0 \le b \le x^-$ that $a \perp b$ and, therefore, $|y| = |a - b| = a + b = u$. Similarly, $c \perp d$, and, therefore, $|z| = v$. $\square$

Theorem 3.2. *Let (x_α) be a net in X such that $x_\alpha \xrightarrow{\text{un}} 0$. Then there exists an increasing sequence of indices (α_k) and a disjoint sequence (d_k) such that $x_{\alpha_k} - d_k \to 0$.*

Proof. Assume first that $x_\alpha \ge 0$ for every α. Pick any α_1. Suppose that $\alpha_1, \ldots, \alpha_{k-1}$ have been constructed. Note that $x_\alpha \wedge x_{\alpha_i} \to 0$ for every $i = 1, \ldots, k-1$. Choose $\alpha_k > \alpha_{k-1}$ so that $\left\| x_{\alpha_k} \wedge x_{\alpha_i} \right\| \le \frac{1}{2^{k+i}}$ for every $i = 1, \ldots, k-1$. This produces an increasing sequence of indices (α_k) such that $\|z_{ik}\| \le \frac{1}{2^{k+i}}$ where $z_{ik} = x_{\alpha_i} \wedge x_{\alpha_k}$, $1 \le i < k$.

For every k, put $v_k = \sum_{i=1}^{k-1} z_{ik} + \sum_{j=k+1}^{\infty} z_{kj}$. Clearly, v_k is defined and $\|v_k\| < \frac{1}{2^k}$. Put $d_k = (x_{\alpha_k} - v_k)^+$. It is easy to see that $0 \le x_{\alpha_k} - d_k \le v_k$, so that $\|x_{\alpha_k} - d_k\| \to 0$ as $k \to \infty$. It is left to show that the sequence (d_k) is disjoint. Let $k < m$. Then

$$
\begin{aligned}
d_k &= (x_{\alpha_k} - v_k)^+ \le (x_{\alpha_k} - z_{km})^+ = x_{\alpha_k} - x_{\alpha_k} \wedge x_{\alpha_m}, \text{ and} \\
d_m &= (x_{\alpha_m} - v_m)^+ \le (x_{\alpha_m} - z_{km})^+ = x_{\alpha_m} - x_{\alpha_k} \wedge x_{\alpha_m}.
\end{aligned}
$$

It follows that $d_k \perp d_m$.

For the general case, we first apply the first part of the proof to the net $\big(|x_\alpha|\big)$ and produce an increasing sequence of indices (α_k) and two positive sequences (w_k) and (h_k) such that $|x_{\alpha_k}| = w_k + h_k$, (w_k) is disjoint, and $h_k \to 0$. By Lemma 3.1, we can find sequences (d_k) and (g_k) in X with $|d_k| = w_k$, $|g_k| = h_k$ and $x_{\alpha_k} = d_k + g_k$. It follows that (d_k) is a disjoint sequence and $g_k \to 0$. Thus, $x_{\alpha_k} - d_k \to 0$.   □

Remark 3.3. Theorem 3.2 is a variant of the Kadeč-Pełczyński dichotomy theorem; cf [LT79, p.38]. Theorem 3.2 clearly implies [GTX, Lemma 6.7]; unlike in [GTX, Lemma 6.7], we do not require the space to be order continuous or the net be norm bounded. Also, we start with a net instead of a sequence.

Recall the following standard fact; see, e.g., Exercise 13 in [AA02, p. 25].

Proposition 3.4. *Every norm convergent sequence in a Banach lattice has a subsequence which converges in order to the same limit.*

Corollary 3.5. *Let (x_α) be a net in an order continuous Banach lattice X such that $x_\alpha \xrightarrow{\text{un}} 0$. Then there exists an increasing sequence of indices (α_k) such that $x_{\alpha_k} \xrightarrow{\text{uo}} 0$ and $x_{\alpha_k} \xrightarrow{\text{un}} 0$.*

Proof. Let (α_k) and (d_k) be as in Theorem 3.2. Since (d_k) is disjoint, we have $d_k \xrightarrow{\text{uo}} 0$ and, therefore, $d_k \xrightarrow{\text{un}} 0$. It now follows from $x_{\alpha_k} - d_k \to 0$ that $x_{\alpha_k} - d_k \xrightarrow{\text{un}} 0$ and, therefore, $x_{\alpha_k} \xrightarrow{\text{un}} 0$. Furthermore, since $x_{\alpha_k} - d_k \to 0$, passing to a further subsequence, we may assume that $x_{\alpha_k} - d_k \xrightarrow{\text{o}} 0$ and, therefore, $x_{\alpha_k} - d_k \xrightarrow{\text{uo}} 0$. This yields $x_{\alpha_k} \xrightarrow{\text{uo}} 0$.   □

Note that Corollary 3.5 provides a partial answer to Question 2.14.

4. Uo-convergent subsequences and convergence in measure

We now have an analogue of Proposition 3.4 for un- and uo-convergences.

Proposition 4.1. *If $x_n \xrightarrow{\text{un}} 0$ then there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{\text{uo}} 0$.*

Proof. Define $e := \sum_{n=1}^\infty \frac{|x_n|}{2^n \|x_n\|}$. Let B_e be the band generated by e in X. It follows from $x_n \xrightarrow{\text{un}} 0$ that $|x_n| \wedge e \to 0$ in X and, therefore, in B_e. There exists a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k}| \wedge e \xrightarrow{\text{o}} 0$ in B_e. Since e is a weak unit in B_e, we have $x_{n_k} \xrightarrow{\text{uo}} 0$ in B_e. Finally, since B_e is an ideal in X, it follows from [GTX, Corollary 3.8] that $x_{n_k} \xrightarrow{\text{uo}} 0$ in X.   □

It was observed in [Tro04, Example 23] that for sequences in $L_p(\mu)$, where μ is a finite measure, un-convergence agrees with convergence in measure. We now provide an alternative proof of this fact.

Corollary 4.2. *([Tro04]) Let (f_n) be a sequence in $L_p(\mu)$ where $1 \le p < \infty$ and μ is a finite measure. Then $f_n \xrightarrow{un} 0$ iff $f_n \xrightarrow{\mu} 0$.*

Proof. Without loss of generality, $f_n \ge 0$ for all n. Suppose $f_n \xrightarrow{\mu} 0$. It is easy to see that $f_n \wedge \mathbb{1} \to 0$ in the norm of $L_p(\mu)$. It follows from Lemma 2.11 that $f_n \xrightarrow{un} 0$.

Conversely, suppose that $f_n \xrightarrow{un} 0$. Then every subsequence (f_{n_k}) is still un-null and, therefore, by Proposition 4.1, has a further subsequence $f_{n_{k_i}}$ such that $f_{n_{k_i}} \xrightarrow{uo} 0$ and, therefore, $f_{n_{k_i}} \xrightarrow{a.e.} 0$. This yields $f_n \xrightarrow{\mu} 0$. $\qquad\square$

Remark 4.3. In the last step of the preceding proof, we used the fact that given a sequence of measurable functions over a measure space with a finite measure, the sequence converges in measure iff every subsequence has a further subsequence which converges a.e. (to the same limit); see, e.g., Corollary 19 in [Roy88, p. 96]. Note that Proposition 4.1 may be viewed as an extension of one of the directions of this equivalence to general Banach lattices. The next result shows that for order continuous Banach lattices the other direction extends as well.

Theorem 4.4. *A sequence in an order continuous Banach lattice X is un-null iff every subsequence has a further subsequence which is uo-null.*

Proof. The forward implication is Proposition 4.1. To show the converse, assume that $x_n \xrightarrow{un} 0$. Then there exist $\delta > 0$, $u \in X_+$, and a subsequence (x_{n_k}) such that $\big\| |x_{n_k}| \wedge u \big\| > \delta$ for all k. By assumption, there is a subsequence $(x_{n_{k_i}})$ of (x_{n_k}) such that $x_{n_{k_i}} \xrightarrow{uo} 0$, and, therefore, $x_{n_{k_i}} \xrightarrow{un} 0$ by Proposition 2.5. This yields $|x_{n_{k_i}}| \wedge u \to 0$, which is a contradiction. $\qquad\square$

Remark 4.5. Again, Example 2.6 shows that the order continuity assumption cannot be removed.

Suppose that X is an order continuous Banach lattice with a weak unit e. It is known that X can be represented as an order and norm dense ideal in $L_1(\mu)$ for some finite measure μ. That is, there is a vector lattice isomorphism $T \colon X \to L_1(\mu)$ such that $\mathrm{Range}\,T$ is an order and norm dense ideal of $L_1(\mu)$. Note that T need not be a norm isomorphism, though T may be chosen to be continuous and $Te = \mathbb{1}$. Moreover, $\mathrm{Range}\,T$ contains $L_\infty(\mu)$ as a norm and order dense ideal.

It is common to identify X with $\operatorname{Range} T$ and just view X as an ideal of $L_1(\mu)$; we also identify e with $\mathbb{1}$. We call such an inclusion of X into an $L_1(\mu)$ space an **AL-representation** of X. We refer the reader to [LT79, Theorem 1.b.14] or [GTX, Section 4] for more details on AL-representations.

It was observed in [GTX, Remark 4.6] that for a sequence (x_n) in X, $x_n \xrightarrow{\text{uo}} 0$ in X iff $x_n \xrightarrow{\text{a.e.}} 0$ in $L_1(\mu)$. We prove an analogous result for un-convergence.

Theorem 4.6. *Let X be an order continuous Banach lattice with a weak unit; let $L_1(\mu)$ be an AL-representation for X. For a sequence (x_n) in X, we have $x_n \xrightarrow{\text{un}} 0$ in X iff $x_n \xrightarrow{\mu} 0$ in $L_1(\mu)$.*

Proof. By Theorem 4.4, $x_n \xrightarrow{\text{un}} 0$ in X iff for every subsequence (x_{n_k}) there is a further subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \xrightarrow{\text{uo}} 0$. The latter is equivalent to $x_{n_{k_i}} \xrightarrow{\text{a.e.}} 0$. Now apply Remark 4.3. $\qquad\square$

5. When do un- and uo-convergences agree?

Our next goal is to prove that uo- and un-convergences for sequences agree iff X is order continuous and atomic. Recall that a non-zero element $a \in X_+$ is an **atom** iff the ideal I_a consists only of the scalar multiples of a. In this case, I_a is a projection band. Let P_a be the corresponding band projection. We say that X is **atomic** or **discrete** if it equals the band generated by all the atoms in it. Suppose that X is atomic, and fix a maximal disjoint collection A of atoms in X. For every $x \in X_+$, we have $x = \bigvee_{a \in A} x_a a$, where $x_a a = P_a x$. One can also write this as $x = \sum_{a \in A} x_a a$, where the sum is understood as the order limit (or the supremum) of sums over finite subsets of A. This sum may be viewed as a coordinate expansion of x over A. Furthermore, if we are also given a positive vector $y = \sum_{a \in A} y_a a$, then $x \le y$ iff $x_a \le y_a$ for every $a \in A$. Suppose now that, in addition, X is order continuous. Then it can be shown that only countably many of coefficients x_a are non-zero. Enumerating them, we get $x = \sum_{i=1}^{\infty} x_{a_i} a_i$, where the series converges in order and, therefore, in norm. For details, see [AB03], Exercise 7 in [Sch74, p. 143], and the proof of Proposition 1.a.9 in [LT79]. We will need two standard lemmas.

Lemma 5.1. *Suppose that X is atomic and order continuous, and (x_n) is an order bounded sequence in X. If $x_n \to 0$ then $x_n \xrightarrow{\text{o}} 0$.*

Proof. Without loss of generality, $x_n \ge 0$ for all n. Let $u \in X_+$ such that $x_n \le u$ for every n. There is a sequence of distinct atoms (a_i) in A such that $u = \sum_{i=1}^{\infty} u_i a_i$ for some coefficients (u_i); the series converges

in norm and in order. Given $n \in \mathbb{N}$, it follows from $0 \leq x_n \leq u$ that we can write $x_n = \sum_{i=1}^{\infty} x_{ni} a_i$ for some sequence of coefficients (x_{ni}). Note that $0 \leq x_{ni} a_i \leq x_n$ for every n and i; it follows that $\lim_n x_{ni} = 0$ for every i; that is, the sequence (x_n) converges to zero "coordinate-wise".

For each $k \in \mathbb{N}$, define

$$v_k = \sum_{i=1}^{k} (\tfrac{1}{k} \wedge u_i) a_i + \sum_{i=k+1}^{\infty} u_i a_i.$$

It is easy to see that $v_k \downarrow 0$. On the other hand, since $x_n \leq u$ and (x_n) converges to zero coordinate-wise, for every k we can find n_k such that $x_n \leq v_k$ whenever $n \geq n_k$. It follows that $x_n \xrightarrow{\text{o}} 0$. $\square$

Lemma 5.2. *If μ is a finite non-atomic measure then there exists a sequence (f_n) in $L_\infty(\mu)$ which converges to zero in measure but not a.e..*

Proof. In the special case when μ is the Lebesgue measure on the unit interval, we take (f_n) to be the "typewriter" sequence

$$f_n = \chi_{[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}]} \text{ where } k \geq 0 \text{ such that } 2^k \leq n < 2^{k+1}.$$

In the general case, we produce a similar sequence using the fact that a non-atomic finite measure space can be partitioned into measurable pieces of arbitrarily small measure; this follows easily from Exercise 2 in [Hal70, p. 174]. $\square$

Theorem 5.3. *The following are equivalent:*

(i) $x_n \xrightarrow{\text{uo}} 0 \iff x_n \xrightarrow{\text{un}} 0$ *for every sequence (x_n) in X;*
(ii) X *is order continuous and atomic.*

Proof. (ii)$\Rightarrow$(i) Suppose X is order continuous and atomic. The implication $x_n \xrightarrow{\text{uo}} 0 \Rightarrow x_n \xrightarrow{\text{un}} 0$ is trivial, and the reverse implication follows immediately from Lemma 5.1.

(i)$\Rightarrow$ (ii) Let (x_n) be a disjoint order bounded sequence in X. Then $x_n \xrightarrow{\text{uo}} 0$ by [GTX, Corollary 3.6]. By assumption, $x_n \xrightarrow{\text{un}} 0$. Since the sequence is order bounded, this yields $x_n \to 0$. Hence, X is order continuous. It follows that every closed ideal in X is a projection band.

It remains to show that X is atomic. Suppose not; then the band X_1 generated by all the atoms in X is a proper subset of X. Let X_2 be the complementary band. Fix a non-zero $w \in X_2^+$, and let $Y = B_w$, the band generated by w in X. Clearly, $Y \subseteq X_2$, Y is order continuous, w is a weak unit in Y, and Y has no atoms. We can find an AL-representation for Y such that $L_\infty(\mu) \subseteq Y \subseteq L_1(\mu)$. Since Y has no atoms, it is easy to see that μ is a non-atomic measure.

By Lemma 5.2, there is a sequence (x_n) in $L_\infty(\mu)$ such that $x_n \xrightarrow{\mu} 0$ but $x_n \xnrightarrow{\text{a.e.}} 0$. It follows that $x_n \xrightarrow{\text{un}} 0$ but $x_n \xnrightarrow{\text{uo}} 0$ in Y and, therefore, in X; a contradiction. $\qquad\square$

6. Un-convergence and weak convergence

In this section, we consider the relationship between un- and weak convergences. For a mononote net, weak convergence implies norm convergence, and, therefore, un-convergence; however, weak convergence does not imply un-convergence in general. For example, the following fact was observed in [CW98, Theorem 2.2]:

Lemma 6.1. ([CW98]) If X is non-atomic and order continuous, and $x \in X_+$, then there exists a sequence (x_n) such that $x_n \xrightarrow{\text{w}} 0$ yet $|x_n| = x$ for all n.

Clearly, (x_n) is not un-null.

Proposition 6.2. The following are equivalent:
(i) $x_n \xrightarrow{\text{w}} 0$ implies $x_n \xrightarrow{\text{un}} 0$ for every sequence (x_n) in X;
(ii) X is order continuous and atomic.

Proof. (i)$\Rightarrow$(ii) The proof is similar to that of Theorem 5.3. Let (x_n) be a disjoint order bounded sequence in X. Then $x_n \xrightarrow{\text{w}} 0$. By assumption, $x_n \xrightarrow{\text{un}} 0$. Since (x_n) is order bounded, this yields $x_n \to 0$. Therefore, X is order continuous.

Now suppose that X is not atomic. Then $X = X_1 \oplus X_2$ where X_1 is the band generated by the atoms, and X_2 is the complementary band. Since X is not atomic, we have $X_2 \neq \{0\}$. Now apply Lemma 6.1 to X_2.

(ii)$\Rightarrow$(i) Lemma 6.14 in [GTX] asserts that if X is atomic and $x_n \xrightarrow{\text{w}} 0$ then $x_n \xrightarrow{\text{uo}} 0$. If, in addition, X is order continuous, this yields $x_n \xrightarrow{\text{un}} 0$. $\qquad\square$

In the previous result, the conditions for a weakly-null sequence to be un-null are quite strong. On the other hand, in an arbitrary Banach lattice, this fact is true for monotone nets. If we remove the constraint that X is atomic, then we obtain the following result. Recall that for an order bounded positive net (x_α) in an order continuous Banach lattice, $x_\alpha \xrightarrow{\text{w}} 0$ implies $x_\alpha \to 0$ by [AB06, Theorem 4.17].

Proposition 6.3. Let (x_α) be a positive net in an order continuous Banach lattice X. If $x_\alpha \xrightarrow{\text{w}} 0$ then $x_\alpha \xrightarrow{\text{un}} 0$.

Proof. For every $u \in X_+$ we have $0 \le x_\alpha \wedge u \le x_\alpha$ for every α. This yields $x_\alpha \wedge u \xrightarrow{\text{w}} 0$. Since this net is positive and order bounded, $x_\alpha \wedge u \to 0$. Hence, $x_\alpha \xrightarrow{\text{un}} 0$. $\qquad\square$

We have now seen several conditions on X that yield $x_n \xrightarrow{\text{w}} 0$ implies $x_n \xrightarrow{\text{un}} 0$. For the converse, we have the following.

Theorem 6.4. *If X^* is order continuous then $x_\alpha \xrightarrow{\text{un}} 0$ implies $x_\alpha \xrightarrow{\text{w}} 0$ for every norm bounded net (x_α) in X_+.*

Proof. Suppose that X^* is order continuous and (x_α) is a norm bounded net in X_+ with $x_\alpha \xrightarrow{\text{un}} 0$. Without loss of generality, $\|x_\alpha\| \le 1$ for every α. Since X^* is order continuous, it follows from [AB06, Theorem 4.19] that for every $\varepsilon > 0$ and every $f \in X_+^*$ there exists $u \in X_+$ such that $f\big(|x| - |x| \wedge u\big) < \varepsilon$ whenever $\|x\| \le 1$. In particular, $f\big(x_\alpha - x_\alpha \wedge u\big) < \varepsilon$ for every α. It follows from $x_\alpha \wedge u \to 0$ that $f(x_\alpha) < \varepsilon$ for all sufficiently large α. Hence, $f(x_\alpha) \to 0$ and, therefore, $x_\alpha \xrightarrow{\text{w}} 0$. $\qquad\square$

We do not know whether the converse is true. If X^* is not order continuous then ℓ_1 is lattice embeddable in X; see, e.g., [AB06, Theorem 4.69]. Let (e_n) be the standard basis of ℓ_1 viewed as a sequence in X. It is easy to see that $e_n \xrightarrow{\text{w}} 0$ in ℓ_1 and, therefore, in X. It is also easy to see that $e_n \xrightarrow{\text{un}} 0$ in ℓ_1. However, this does not imply that $e_n \xrightarrow{\text{un}} 0$ in X; see Example 2.7.

7. Un-convergence is topological

It is well known that a.e. convergence is not topological; see, e.g., [Ord66]. That is, this convergence is not given by a topology. It follows that uo-convergence need not be topological in general. We show that un-convergence is topological. Moreover, we explicitly define the neighborhoods of this topology.

Given an $\varepsilon > 0$ and a non-zero $u \in X_+$, we put

$$V_{u,\varepsilon} = \big\{ x \in X \ : \ \big\| |x| \wedge u \big\| < \varepsilon \big\}.$$

Let $\mathcal{N}_0$ be the collection of all the sets of this form. We claim that $\mathcal{N}_0$ is a base of neighborhoods of zero for some Hausdorff linear topology. Once we establish this, we will be able to define arbitrary neighborhoods as follows: a subset U of X is a neighbourhood of y if $y + V \subseteq U$ for some $V \in \mathcal{N}_0$. It follows immediately from the definition of un-convergence that $x_\alpha \xrightarrow{\text{un}} 0$ iff every set in $\mathcal{N}_0$ contains a tail of this net, hence the un-convergence is exactly the convergence given by this topology.

We have to verify that $\mathcal{N}_0$ is indeed a base of neighborhoods of zero; cf. [KN76, Theorem 5.1] or [Run05, Theorem 3.1.10].

First, every set in $\mathcal{N}_0$ trivially contains zero.

Second, we need to show that the intersection of any two sets in $\mathcal{N}_0$ contains another set in $\mathcal{N}_0$. Take V_{u_1,ε_1} and V_{u_2,ε_2} in $\mathcal{N}_0$. Put $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$ and $u = u_1 \vee u_2$. We claim that $V_{u,\varepsilon} \subseteq V_{u_1,\varepsilon_1} \cap V_{u_2,\varepsilon_2}$. Indeed, take any $x \in V_{u,\varepsilon}$. Then $\big\| |x| \wedge u \big\| < \varepsilon$. It follows from $|x| \wedge u_1 \le |x| \wedge u$ that

$$\big\| |x| \wedge u_1 \big\| \le \big\| |x| \wedge u \big\| < \varepsilon \le \varepsilon_1,$$

so that $x \in V_{u_1,\varepsilon_1}$. Similarly, $x \in V_{u_2,\varepsilon_2}$.

It is easy to see that $V_{u,\varepsilon} + V_{u,\varepsilon} \subseteq V_{u,2\varepsilon}$. This immediately implies that for every U in $\mathcal{N}_0$ there exists $V \in \mathcal{N}_0$ such that $V + V \subseteq U$. It is also easy to see that for every $U \in \mathcal{N}_0$ and every scalar λ with $|\lambda| \le 1$ we have $\lambda U \subseteq U$.

Next, we need to show that for every $U \in \mathcal{N}_0$ and every $y \in U$, there exists $V \in \mathcal{N}_0$ such that $y + V \subseteq U$. Let $y \in V_{u,\varepsilon}$ for some $\varepsilon > 0$ and a non-zero $u \in X_+$. We need to find $\delta > 0$ and a non-zero $v \in X_+$ such that $y + V_{v,\delta} \subseteq V_{u,\varepsilon}$. Put $v := u$. It follows from $y \in V_{u,\varepsilon}$ that $\big\| |y| \wedge u \big\| < \varepsilon$; take $\delta := \varepsilon - \big\| |y| \wedge u \big\|$. We claim that $y + V_{v,\delta} \subseteq V_{u,\varepsilon}$. Let $x \in V_{v,\delta}$; it suffices show that $y + x \in V_{u,\varepsilon}$. Indeed, $|y + x| \wedge u \le |y| \wedge u + |x| \wedge u$, so that

$$\big\| |y + x| \wedge u \big\| \le \big\| |y| \wedge u \big\| + \big\| |x| \wedge u \big\| < \big\| |y| \wedge u \big\| + \delta = \varepsilon.$$

Finally, in order to show that the topology is Hausdorff, we need to verify that $\bigcap \mathcal{N}_0 = \{0\}$. Indeed, suppose that $0 \ne x \in V_{u,\varepsilon}$ for all non-zero $u \in X_+$ and $\varepsilon > 0$. In particular, $x \in V_{|x|,\varepsilon}$, so that $\|x\| = \big\| |x| \wedge |x| \big\| < \varepsilon$ for every $\varepsilon > 0$; a contradiction.

Note that we could also conclude that the topology is linear and Hausdorff from Lemma 2.1.

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